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TOPICAL REVIEW

Nonlinear aspects of the renormalization group flows of Dyson's hierarchical model

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Abstract

We review recent results concerning the renormalization group (RG) transformation of Dyson's hierarchical model (HM). This model can be seen as an approximation of a scalar field theory on a lattice. We introduce the HM and show that its large group of symmetry simplifies drastically the blockspinning procedure. Several equivalent forms of the recursion formula are presented with unified notations. Rigourous and numerical results concerning the recursion formula are summarized. It is pointed out that the recursion formula of the HM is inequivalent to both Wilson's approximate recursion formula and Polchinski's equation in the local potential approximation (despite the very small difference with the exponents of the latter). We draw a comparison between the RG of the HM and functional RG equations in the local potential approximation. The construction of the linear and nonlinear scaling variables is discussed in an operational way. We describe the calculation of non-universal critical amplitudes in terms of the scaling variables of two fixed points. This question appears as a problem of interpolation between these fixed points. Universal amplitude ratios are calculated. We discuss the large-N limit and the complex singularities of the critical potential calculable in this limit. The interpolation between the HM and more conventional lattice models is presented as a symmetry breaking problem. We briefly introduce models with an approximate supersymmetry. One important goal of this review is to present a configuration space counterpart, suitable for lattice formulations, of functional RG equations formulated in momentum space (often called exact RG equations and abbreviated ERGE).

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Contents

Frequently used abbreviations

RG:	renormalization group
ERGE:	exact renormalization group equations
LPA:	local potential approximation
HM:	(Dyson's) Hierarchical model
HT:	high-temperature
UV:	Ultra-violet
IR:	Infra-red
ℓ:	the change in linear scale after one RG transformation
Φ:	the sum of all the fields
Ω:	the total number of sites
ϕ_c:	the classical field or magnetization density
ϕ_n:	the sum of the fields in a block of size 2^n
W_n	the "raw" local measure after blockspinning 2^n sites
$\mathcal{W}_n$:	the rescaled local measure after n RG transformations
$\mathcal{H}_n$:	Baker's form of $\mathcal{W}_n$

$\mathcal{G}_n$: the Gallavotti's form of $\mathcal{W}_n$
$c \equiv 2^{1-2/D}$

1. Introduction

Quantum field theorists face the arduous task of figuring out the large-scale implications of models defined by interactions at a small scale. In general, the large distance behaviour of the theory can be encoded in an effective action S_{eff} which describes the interactions of the zero-momentum modes of the fields. Knowing S_{eff}, we can answer important questions regarding, for instance, the stability and triviality of the theory. The renormalization group (RG) method [1–4] was designed to calculate S_{eff} by a sequence of small steps where the high energy modes are integrated progressively. This procedure generates a sequence of $S_{\text{eff},\Lambda}$, where Λ is the scale above which we have integrated all the modes. In the process, many new interactions are introduced and with them also functions evolving with Λ. This can be seen as a flow in the space of theories. We call this type of flows the RG flows. Important simplifications may occur near fixed points that have only a few unstable directions, and universal properties may emerge. Despite the conceptual beauty of the construction, the practical calculation of the RG flows remains very difficult. The behaviour near fixed points can usually be handled by some expansion (weak coupling, strong coupling, etc . . .), but interpolating between two fixed points is in general a major difficulty. Unfortunately, this is essential to calculate S_{eff}.

In the following, we review our present understanding of the interpolation between fixed points for Dyson's hierarchical model [5, 6], a model for which the calculation of $S_{\text{eff},\Lambda}$ can be reduced to the calculation of the effective potential $V_{\text{eff},\Lambda}$ and can be performed numerically with great accuracy. Other type of hierarchical models have been discussed in the literature (see, for instance, [7]), however, in the following, we only consider Dyson's model. The hierarchical model (HM) will only refer to Dyson's model in this review. The RG transformation for the HM can be expressed as a simple one-variable integral equation very similar to the so-called approximate recursion formula proposed by Wilson [3]. In many respects, the techniques involved in the solution of this model can be compared to those used in elementary quantum mechanics. The reason for this remarkable simplicity is that the kinetic term of the model is not renormalized. In other words, it is a model for which the local potential approximation (LPA) is exact.

In recent years, the LPA has been widely used in the context of exact renormalization group equations (ERGE) and has generated a lot of interest. We recommend references [8–10] for reviews of the recent progress. ERGE allow in principle the study of global and nonlinear aspects of the RG flows of field theoretical or statistical models. However, truncation methods are necessary in order to make practical calculations. One particularly popular choice is to combine Polchinski's ERGE [11] with the LPA [12]. This results in a simple partial differential equation (called 'Polchinski's equation' below) for the effective potential. Polchinski's equation can also be obtained [13] as an infinitesimal version of the RG equation for the HM. This suggests that the linearized theories should be close to each other. Accurate calculations of the critical exponents [14–19] show that the exponents differ only in the fifth significant digit. In [18], it was believed that the two exponents should coincide, however this is not the case (this will be explained in section 9).

Note that the word 'exact' in ERGE is sometimes a source of controversy. First, the exact equations are often used together with approximations. Second, it somehow implies that other equations are not exact. The terminology appears in the 1974 review of Wilson and Kogut [4]. In this context, it is perfectly correct since it appears after several approximate

equations have been introduced. Nowadays, the terminology 'functional RG equations' may be more appropriate, since the equations describe the response of generating functionals to a small change in the cutoff. These equations make sense independently of perturbation theory. Other RG equations express the invariance of perturbative series under a change in the renormalization scale. They are also exact, but they are meant to be used in a more restricted framework. Despite these remarks, we will use the abbreviation ERGE in the rest of this review since it has become standard in the literature, but the first 'E' should be taken with a grain of salt.

Originally, the HM was introduced by Dyson [5] as a long-range ferromagnetic Ising model with couplings weaker than the one-dimensional Ising model with long-range couplings falling off like a power. The existence of a phase transition and of the infinite volume limit for some range of the parameter controlling the decay of the interactions can be proved rigorously. Historical motivations and rigourous results are discussed in more detail in section 6. The model was rediscovered a few years later by Baker [6] in an attempt to construct models for which Wilson's approximate recursion formula or an integral equation of the same form become the exact RG transformation. In this context, the HM appears rather like an approximation of a scalar field theory on a D-dimensional lattice. Later, we call the process of interpolation between the HM and lattice models with nearest neighbour interaction the 'improvement of the hierarchical approximation'.

One important goal of the review is to stress the similarities between the Polchinski equation (a generic computational tool) and the HM (often perceived as a toy model). The motivations for using the HM model are as follows.

- It is a lattice model right from the beginning.
- High-accuracy methods exist to calculate numerically the critical exponents and the RG flows for arbitrary potentials.
- Conventional expansions (weak and strong coupling, ϵ-expansion) can be implemented easily and large-order series can be obtained.
- The hierarchical approximation can in principle be improved [20, 21].

The study of critical phenomena has reached a stage where many methods have been refined to a point where they provide numbers very close to each other [22, 23]. In the case of the HM, all the approximations can be compared to very accurate numerical answers. The ability of constructing the RG flows very accurately means that we can study general features of these flows far away from fixed points. This type of study is also possible using the LPA [8, 24–26].

An interesting feature of the HM is its discrete scale invariance which, depending on the context, can be seen either as an annoyance [27, 28] or an interesting intrinsic property [29]. The review is focused on doing calculations directly in three dimensions. The ϵ-expansion near four dimensions,which is reviewed in [30], is not discussed here.

The paper is organized as follows. In section 2, we briefly review the basic concepts of the RG method, the scaling hypothesis and the practical difficulties of blockspinning. This section motivates approximations that simplify the blockspinning procedure. Dyson's HM is introduced in section 3, with the notation used later in the review. Several equivalent forms of the recursion formula are presented in section 4. More general recursion formulae which coincide with the HM's one for a particular choice of what we call the scale parameter are introduced in section 5. This includes Wilson's approximate recursion formula [3] which is inequivalent to the recursion formula of the HM. In the following, we denote the scale parameter ℓ. In the literature, the same parameter is often denoted L, but we preferred to keep that symbol for the linear size of the whole system.

Having introduced the basic concepts, motivations for the HM, numerical and rigourous results are reviewed in section 6. The numerical treatment of the recursion formula is discussed in section 7 and perturbation theory with a large field cutoff in section 8. The relation to Polchinski's ERGE in the LPA is discussed in section 9.

We introduce the nonlinear scaling variables associated with a fixed point in section 10. These quantities, originally introduced by Wegner [31], transform multiplicatively under a RG transformation. They have features similar to the action-angle variables used in classical mechanics. Small-denominator problems are in general present and need to be discussed in each case. In section 11, we describe the non-universal critical amplitudes as RG invariants made out of two nonlinear scaling fields. A practical calculation based on this method requires the ability of using both sets of scaling fields in a common intermediate region, in other words, to interpolate between the fixed points.

The notion of a nontrivial continuum limit, originally introduced by Wilson [32], is reviewed and applied to the HM in section 12. Calculations of critical amplitudes and their universal ratios are then discussed. In section 13, we introduce the extension of the HM for N components and discuss the large-N limit. We compare the results with those obtained with the Polchinski equation emphasizing the difference in the fifth digit already mentioned above, which reflects the non-equivalence of the models.

The possibility of improving the hierarchical approximation by breaking its symmetries in a systematic way is discussed in section 14. Compared to the improvement of the LPA by the derivative expansion, the improvement of the hierarchical approximation is an underdeveloped subject. On the other hand, it is clear that much progress remains to be done in the ERGE approach in order to match the accuracy of other methods [23, 33, 34] for the calculation of the critical exponents [35]. We hope that this review will facilitate the communication between the two approaches. Reference [36] is a very recent example of progress made in this direction. Finally, the possibility of including fermions in approximately supersymmetric models is briefly discussed in section 15.

One motivation to study global and nonlinear aspects of RG flows not covered in this review is to improve our understanding of quantum chromodynamics. In this theory of strongly interacting quarks and gluons, weakly interacting particles are seen at short distance (asymptotic freedom), while nonperturbative effects cause the confinement of quarks and gluons at a large distance. Understanding how these two behaviour can be smoothly connected is a single theory amounts to interpolate between two fixed points of the renormalization group (RG) transformation. Despite recent progress [10, 37–42] it remains a challenge to understand confinement in terms of weak-coupling variables. This is work for the future.

2. General and practical aspects of the RG method

In this section, we present the basic ideas behind the RG method and point out the practical difficulties associated with the so-called blockspin method.

2.1. Statement of the problem

We are interested in the large distance (low momentum, long wavelength) behaviour of scalar models. We consider a generic scalar model with a lattice regularization and an action S. The scalar field ϕ_x is coupled linearly to a constant source J. We call the total field

$$\Phi = \sum_x \phi_x, \tag{2.1}$$

and the total number of sites Ω. With these notations, the partition function reads

$$Z[J] = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_x \mathrm{d}\phi_x \exp(-S + \Phi J). \tag{2.2}$$

We define the average at $J = 0$ of an arbitrary function A of the fields as

$$\langle A \rangle = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_x \mathrm{d}\phi_x A \exp(-S)/Z[0] \tag{2.3}$$

It is clear that

$$Z[J]/Z[0] = 1 + \sum_{q=1}^{\infty} \frac{1}{q!} J^q \langle \Phi^q \rangle. \tag{2.4}$$

The connected parts can be obtained by taking the logarithm of this expression. At the lowest orders,

$$\begin{aligned} \langle \Phi^2 \rangle^c &= \langle \Phi^2 \rangle - (\langle \Phi \rangle)^2 \\ \langle \Phi^3 \rangle^c &= \langle \Phi^3 \rangle - 3\langle \Phi^2 \rangle \langle \Phi \rangle + 2(\langle \Phi \rangle)^3 \\ \langle \Phi^4 \rangle^c &= \langle \Phi^4 \rangle - 4\langle \Phi^3 \rangle \langle \Phi \rangle - 3(\langle \Phi^2 \rangle)^2 + 12\langle \Phi^2 \rangle (\langle \Phi \rangle)^2 - 6(\langle \Phi \rangle)^4. \end{aligned} \tag{2.5}$$

In general, we expect that

$$\langle \Phi^q \rangle^c \propto \Omega, \tag{2.6}$$

and we define

$$\chi^{(q)} = \langle \Phi^q \rangle^c / \Omega \tag{2.7}$$

even though the individual terms scale faster than Ω. Unless we take the infinite volume limit, we should in principle write $\chi_\Omega^{(q)}$ in order to remind the dependence on the volume. In the rest of this subsection, this dependence will be kept implicit. We then write the generating function of the connected densities

$$(1/\Omega) \ln(Z[J]/Z[0]) = \sum_{q=1}^{\infty} \frac{1}{q!} J^q \chi^{(q)}. \tag{2.8}$$

It is common to call

$$G[J] \equiv -(1/\Omega) \ln(Z[J]/Z[0]), \tag{2.9}$$

the Gibbs potential and it is clear that

$$\chi^{(q)} = -\partial^q G[J]/\partial J^q. \tag{2.10}$$

We now introduce 1 in the functional integral in the following way:

$$1 = \int_{-\infty}^{+\infty} \mathrm{d}\phi_c \delta(\phi_c - \Phi/\Omega). \tag{2.11}$$

The partition function becomes [43]

$$Z[J] = \int_{-\infty}^{+\infty} \mathrm{d}\phi_c \exp(-\Omega(V_{\mathrm{eff}}(\phi_c) - \phi_c J)), \tag{2.12}$$

with

$$\exp(-\Omega V_{\mathrm{eff}}(\phi_c)) \equiv \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_x \mathrm{d}\phi_x \exp(-S) \delta(\phi_c - \Phi/\Omega). \tag{2.13}$$

If for any J, $V_{\rm eff}(\phi_c)-\phi_c J$ has a unique minimum at $\phi_c = \bar{\phi}_c$, we have in the limit of arbitrarily large Ω, that

$$Z[J] \propto \exp(-\Omega(V_{\rm eff}(\bar{\phi}_c) - \bar{\phi}_c J)). \tag{2.14}$$

Here $\bar{\phi}_c$ is a function of J defined implicitly by

$$\partial V_{\rm eff}(\phi_c)/\partial\phi_c|_{\phi_c=\bar{\phi}_c} = J. \tag{2.15}$$

Here $G[J]$ and $V_{\rm eff}(\phi_c)$ are related by a Legendre transform

$$G[J] = V_{\rm eff}(\bar{\phi}_c) - \bar{\phi}_c J. \tag{2.16}$$

In the ferromagnetic language, ϕ_c is called the magnetization and J the magnetic field. In analogy with the gas–liquid transition, $V_{\rm eff}$ plays the role of the Helmholtz potential, $\bar{\phi}_c$ plays the role of minus the volume and J the role of the pressure. When we are in the gas phase below the critical temperature (but above the triple point temperature), if we increase the pressure keeping the temperature fixed, the volume decreases until a critical pressure is reached where the gas and the liquid can coexist at an equilibrium pressure but with different specific volumes. The discontinuity in the magnetization is analogue to the specific volume discontinuity. The sign is justified by the fact that as we increase J, the magnetization increases, but if we increase the pressure, the volume decreases.

In the simple case where $V_{\rm eff}$ is an even function with a unique minimum at zero, we can expand

$$V_{\rm eff}(\phi_c) = \sum_{q=1}^{\infty} \frac{1}{2q!}\phi_c^{2q}\Gamma^{(2q)}. \tag{2.17}$$

Using the derivative of this expansion to express J in terms of $\bar{\phi}_c$, plugging into equation (2.16), using equation (2.8) and solving order by order in $\bar{\phi}_c$, we obtain the well-known relations

$$\begin{aligned}
\Gamma^{(2)} &= 1/\chi^{(2)},\\
\Gamma^{(4)} &= -\chi^{(4)}/(\chi^{(2)})^4,\\
\Gamma^{(6)} &= 10(\chi^{(4)})^2/(\chi^{(2)})^7 - \chi^{(6)}/(\chi^{(2)})^6,
\end{aligned} \tag{2.18}$$

and so on. The calculation of the $V_{\rm eff}$ is an important objective the RG method.

2.2. Basic aspects of the RG method

We now consider a scalar model on a D-dimensional cubic lattice with a lattice spacing a. The RG transformation proceeds in two steps. First, we integrate the ℓ^D fields in blocks of linear size ℓa while keeping the sum of the fields in the block constant. We then divide the sums of the fields by a factor $\ell^{(2+D-\eta)/2}$ and treat them as our new field variables. The exponent η is introduced in order to keep the canonical form of the kinetic term and its calculation is nontrivial. We then obtain a new theory in terms of a new field variable which is equivalent to the previous one as long as we only consider processes involving energies smaller than the new ultraviolet cutoff $\sim 1/(\ell a)$. Given an original action, we assume that the RG transformation provides a new effective action expressed in terms of the new field variable. We postpone the discussion of the practical aspects of the partial integration to subsection 2.3.

The information that is kept during the RG transformation is encoded in the average values of all the integer powers of the sum of the fields in the blocks. We call these average values the 'zero momentum Green's functions at finite volume'. This set of values can be thought of as an element of an infinite vector space. In the following we call the trajectories in this

space the RG flows. If we start with an infinite volume lattice, the RG transformation can in principle be repeated an infinite number of times. If this can be done, the resulting effective theory is limited to the zero-momentum Green's functions. We are particularly interested in finding the fixed points of the RG transformation, since the above limit can be simplified in the neighbourhood of a fixed point.

The RG transformation has several obvious fixed points. If the interactions are limited to quadratic ones and there is no restriction on the range of the fields, the partition function can be calculated exactly using diagonalization and Gaussian integration. The model describes non-interacting particles of a given mass. We call this mass in cutoff units m. It can be determined in cutoff units using the zero-momentum two-point function.

After one RG transformation, the mass for the new effective theory is $m\ell$. The Gaussian fixed point (also called the trivial fixed point) corresponds to the theory with $m = 0$. There are usually other fixed points which correspond to stable phases, for instance, the high-temperature fixed point which can be thought as an infinitely massive theory where the fluctuations about the zero field value are entirely suppressed.

In addition to these obvious fixed points, we expect nontrivial fixed points when $D < 4$. These are characterized by one or more unstable directions. The critical hypersurface is given as the stable manifold (e.g. the basin of attraction) of this nontrivial fixed point. Its codimension is the number of unstable directions. In the rest of this section, we specialize the discussion to the case where there is only one unstable direction. The stable manifold can then be reached by considering a family of models indexed by a parameter which can be tuned in order to cross the stable manifold. In field theory context, we usually pick the bare mass to accomplish this purpose. In the statistical mechanics formulation, the inverse temperature β can be tuned to its critical value β_c which is a function of the other interactions.

Near the nontrivial fixed point, we can use the eigenvectors of the linearized RG transformation as a basis. As far as we are close to the fixed point, the average values of the powers of the *rescaled* total field stay approximately unchanged after one transformation. However, at each iteration, the components in the eigendirections are multiplied by the corresponding eigenvalue. In the following, we denote the eigenvalue larger than 1 as λ_1. As we assume that there is only one unstable direction, there is only one such eigenvalue and it is the largest one.

After repeating the renormalization group transformation n times, we have replaced ℓ^{Dn} sites by one site and associated a block variable with it. We define the finite volume susceptibility $\chi_n^{(2)}$ as the average value of the square of the sum of all the (unrescaled) fields inside the block divided by the number of sites ℓ^{Dn}. We can estimate $\chi_n^{(2)}$ near the nontrivial fixed point. The average of the square of the rescaled variables is approximately a constant that we call K_1. To get the susceptibility, we need to go back to the original variables (so we multiply K_1 by $\ell^{(2+D-\eta)n}$) and divide by the volume ℓ^{Dn}. We can also take into account the motion along the unstable direction in the linear approximation. In summary, near the fixed point

$$\chi_n^{(2)} \simeq \ell^{n(2-\eta)}\big(K_1 + K_2\lambda_1^n(\beta_c - \beta)\big). \tag{2.19}$$

The constant K_2 depends on the way the critical hypersurface is approached when β is varied close to β_c. Equation (2.19) is valid only if the linearization procedure is applicable, in other words if $\lambda_1^n(\beta_c - \beta) \ll 1$. On the other hand, when n reaches some critical value $n^\star$ such that

$$\lambda_1^{n^\star}(\beta_c - \beta) \simeq 1, \tag{2.20}$$

nonlinear effects become important and the sign of $(\beta_c - \beta)$ becomes important. In the following, we consider the case of the symmetric phase ($\beta < \beta_c$) which is simpler. The order of magnitude of χ starts stabilizing when n gets of the order of $n^\star$ and

$$\chi_\infty^{(2)} \approx \ell^{n^\star(2-\eta)} \approx (\beta_c - \beta)^{-\gamma}, \tag{2.21}$$

with

$$\gamma = (2 - \eta)\nu, \tag{2.22}$$

and

$$\nu = \ln \ell / \ln \lambda_1. \tag{2.23}$$

When $n \gg n^\star$, the trajectories fall into the completely attractive HT fixed point. The approach of the fixed point is characterized by corrections proportional to negative powers of the volume.

We can also take into account the corrections due to the so-called irrelevant directions which correspond to eigenvalues less than 1 and eigenvectors along the stable manifold. We call the largest of the irrelevant eigenvalues $\lambda_2 < 1$. As n increases, the distance along the corresponding eigenvector shrinks as λ_2^n. The relative size of the correction should be proportional to

$$\lambda_2^{n^\star} \approx (\beta_c - \beta)^{\Delta_s}, \tag{2.24}$$

with

$$\Delta_s = -\ln \lambda_2 / \ln \lambda_1. \tag{2.25}$$

Here Δ_s has a lower index s which is short for subleading and should not be confused with the gap exponent that will be denoted Δ_g. In summary, in the large volume limit and for $\beta \to \beta_c^-$, we have the parametrization

$$\chi^{(2)} \simeq (\beta_c - \beta)^{-\gamma}[A_0 + A_1(\beta_c - \beta)^{\Delta_s} + \cdots]. \tag{2.26}$$

Linearized calculations near the nontrivial fixed point allow us to calculate the exponents γ and Δ_s, but not the amplitudes A_0 and A_1.

It should be emphasized that often the lattice spacing does not appear explicitly but is considered as a function of the bare parameters. The relation between the two is usually established by relating lengths calculated in lattice spacing to physical lengths. The continuum limit consists in taking a trajectory in the space of bare parameters which corresponds to the limit of zero lattice spacing.

When $\beta > \beta_c$, we are in the low-temperature phase and $\chi^{(1)} \neq 0$. The sign of $\chi^{(1)} \neq 0$ is positive (negative) if we take the limit $J \to 0$ by positive (negative) values. Additional subtractions are then required as shown in equation (2.5). The case of the low temperature will be discussed in subsection 7.5 for the HM.

Relations among exponents can be obtained from the scaling hypothesis [44–46]. Making explicit the dependence of the non-analytical part of the Gibbs potential, denoted G_s, on the reduced temperature $t \equiv (\beta_c/\beta) - 1$, the scaling hypothesis amounts to have

$$G_s[\ell^{\mathcal{Y}_t} t, \ell^{\mathcal{Y}_h} J] = \ell^D G_s[t, J]. \tag{2.27}$$

The exponents $\mathcal{Y}$ are connected to the critical exponents discussed above by the relations

$$\mathcal{Y}_t = \frac{1}{\nu} \qquad \mathcal{Y}_h = \frac{D + 2 - \eta}{2}. \tag{2.28}$$

It is also common to introduce exponents which describe the correlations as a function of the inverse distance for the conjugated variables at criticality, namely

$$x_h \equiv D - \mathcal{Y}_h \tag{2.29}$$

$$x_t \equiv D - \mathcal{Y}_t, \tag{2.30}$$

for the field and for the energy respectively. The scaling hypothesis will be further discussed in subsections 3.3, 7.4 and 7.5.

2.3. Practical aspects of blockspinning

In the previous subsection, we have assumed that it was possible to integrate over the variables in a block while keeping the sum of the field constant. This procedure is usually called 'blockspinning', an idea which can be traced back to Kadanoff [47]. In practice, blockspinning is usually quite complicated. However, for models with actions quadratic in the fields, it is possible to do it analytically. Of course, for such models, all the correlation functions can be calculated exactly, but the procedure can be considered as the first step in a perturbative expansion. To fix the ideas, we can consider a one-dimensional lattice model where the Fourier transform of the two-point function (the propagator) is $G(k)$. Assuming that the lattice sites are labelled by integers, we have the periodicity $G(k + 2\pi) = G(k)$. If we now partition the lattice into blocks of even–odd pairs of neighbour sites and blockspin within these blocks, we obtain a new two-point function on a new lattice with a lattice spacing twice larger. If we denote the Fourier transform of the two-point function after n steps $G_n(k)$, the iteration formula is

$$G_{n+1}(k) = (1 + \cos(k/2))G_n(k/2) + (1 - \cos(k/2))G_n(k/2 + \pi). \tag{2.31}$$

It is clear that the 2π periodicity is preserved and no sharp edges are introduced at least in a finite number of iterations. The construction can be extended in arbitrary dimensions and was used as the starting point for the finite-lattice approximation [48].

In order to get an idea of the difficulty to extend the procedure when higher order interactions are introduced, it is instructive to consider the simple case of a 4-sites ring with nearest neighbour quadratic interactions and local quartic interactions:

$$Z = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathrm{d}\phi_1\, \mathrm{d}\phi_2\, \mathrm{d}\phi_3\, \mathrm{d}\phi_4\, \mathrm{e}^{-S_2^A - S_2^B - S_4}, \tag{2.32}$$

with nearest neighbour quadratic terms

$$S_2^A = (\phi_1 - \phi_2)^2 + (\phi_3 - \phi_4)^2, \tag{2.33}$$

and

$$S_2^B = (\phi_2 - \phi_3)^2 + (\phi_4 - \phi_1)^2, \tag{2.34}$$

and local quartic interactions

$$S_4 = \phi_1^4 + \phi_2^4 + \phi_3^4 + \phi_4^4. \tag{2.35}$$

For further convenience, we also define the next to nearest neighbour quadratic interactions

$$S_2^{NNN} = (\phi_1 - \phi_3)^2 + (\phi_2 - \phi_4)^2. \tag{2.36}$$

We now try to blockspin. We pick (1, 2) and (3, 4) as our basic blocks and then combine them into the block containing all the sites. More specifically, it consists in introducing 1 in the integral in the following form:

$$\begin{aligned} 1 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}\Phi\, \mathrm{d}\Phi_{(1,2)}\, \mathrm{d}\Phi_{(3,4)} \delta(\Phi - \Phi_{(1,2)} - \Phi_{(3,4)}) \\ \times \delta(\Phi_{(1,2)} - \phi_1 - \phi_2)\delta(\Phi_{(3,4)} - \phi_3 - \phi_4). \end{aligned} \tag{2.37}$$

If we can perform the integration over $\phi_1 - \phi_2$ and $\phi_3 - \phi_4$, and then over $\Phi_1 - \Phi_2$, we will be able to write

$$Z = \int_{-\infty}^{+\infty} \mathrm{d}\Phi \, \mathrm{e}^{-S_{\mathrm{eff}}(\Phi)}. \tag{2.38}$$

In this illustrative model, $S_{\mathrm{eff}}(\Phi)$ is the main quantity of interest. Because of the quadratic terms S_2^B, we cannot perform the integral over $\phi_1 - \phi_2$ independently of the integral over $\phi_3 - \phi_4$, and blockspinning is potentially more difficult than evaluating the integral without intermediate steps. It is interesting to note that the terms of S_2^B connect the fields across the blocks and are not invariant under independent interchanges $(1 \leftrightarrow 2)$ and $(3 \leftrightarrow 4)$. These two transformations are of order 2 and commute; they generate a group of order 4. If we average S_2^B over the four elements g of this group, we obtain

$$(1/4) \sum_g S_2^B(g) = (1/2)\big(S_2^B + S_2^{NNN}\big). \tag{2.39}$$

If we replace S_2^B by this average, we have twice more terms but with half of the strength. Also, the new terms have a longer range. At first sight, it is not clear that the situation is better than before. However, if we combine this average with one half of the harmless S_2^A, which is invariant under the above mentioned symmetry group, we obtain

$$(1/2)\big(S_2^A + S_2^B + S_2^{NNN}\big) = 4\big(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\big) - (\phi_1 + \phi_2 + \phi_3 + \phi_4)^2. \tag{2.40}$$

The first term affects only the local measure and the second can be incorporated directly into $S_{\mathrm{eff}}(\Phi)$. This simple example illustrates how a symmetry can be used as a guide to build a (modified) model where blockspinning is feasible.

Dyson's hierarchical model is a model where a hierarchical exchange symmetry among the sites is built-in and allows us to blockspin the partition function by performing a sequence of one-dimensional integrals.

3. Dyson's hierarchical model

3.1. The model

In this subsection, we describe Dyson's hierarchical model with the notations used in most of the rest of this review. The relationship between this formulation and other ones found in the literature is discussed in sections 4 and 5. The model requires $2^{n_{\max}}$ sites. We label the sites with $n_{\max}$ indices $x_{n_{\max}}, \ldots, x_1$, each index being 0 or 1. We divide the $2^{n_{\max}}$ sites into two blocks, each containing $2^{n_{\max}-1}$ sites. If $x_{n_{\max}} = 0$, the site is in the first block, if $x_{n_{\max}} = 1$, the site is in the second block. Repeating this procedure n times (for the two blocks, their respective two sub-blocks, etc.), we obtain an unambiguous labelling for each of the sites. The indices on the left provide the coarser division, while the indices on the right provide a finer division. With an appropriate choice of origin, the indices can be interpreted as the binary representation of the site numbers. This is represented graphically in figure 1.

The nonlocal part of the total energy reads

$$H = -\frac{1}{2} \sum_{n=1}^{n_{\max}} \left(\frac{c}{4}\right)^n \sum_{x_{n_{\max}}, \ldots, x_{n+1}} \left(\sum_{x_n, \ldots, x_1} \phi_{(x_{n_{\max}}, \ldots, x_1)} \right)^2. \tag{3.1}$$

The index n, referred to as the 'level of interaction' hereafter, corresponds to the interaction of the total field in blocks of size 2^n. The constant c is a free parameter assumed positive and which controls the decay of the iterations with the size of the blocks and can be adjusted

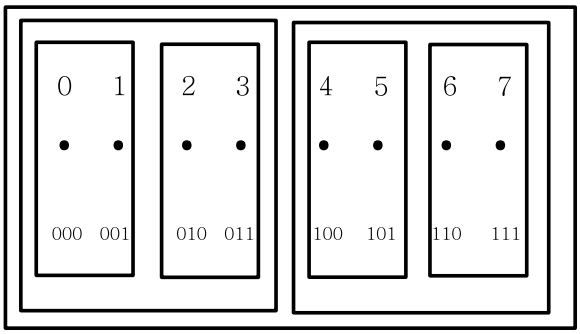

Figure 1. Division in blocks for eight sites. The lower row is $x_3x_2x_1$.

in order to mimic a D-dimensional model. This point is discussed in more detail below (see equation (3.23)).

The field $\phi_{(x_{n_{\max}},\ldots,x_1)}$ is integrated over a local measure which we need to specify. In the following, we will often work with the Ising measure, $W_0(\phi) = \delta(\phi^2 - 1)$ or a Landau–Ginsburg measure of the form $W_0(\phi) = \exp\left(-\frac{1}{2}m^2\phi^2 - \lambda\phi^4\right)$.

In the case of the Ising measure, the only free parameter is c. If all the $\phi = +1$, then the cost in energy for flipping one spin is $2\sum_{l=1}(c/4)^l(2^l - 1)$ and is finite in the infinite volume limit only if $c < 2$. In the following, it will be assumed that $0 < c < 2$.

The hierarchical structure of equation (3.1) allows us to integrate iteratively the fields while keeping their sums in blocks with two sites constant

$$W_{n+1}(\phi_{(1,2)}) = \exp\left((\beta/2)(c/4)^{n+1}\phi_{(1,2)}^2\right) \times \int_{-\infty}^{+\infty} \mathrm{d}\xi\, W_n\left(\frac{\phi_{(1,2)}}{2} + \xi\right) W_n\left(\frac{\phi_{(1,2)}}{2} - \xi\right), \tag{3.2}$$

where $\phi_{(1,2)}$ is understood as the sum of the two fields in the block. After $n_{\max}$ integrations, we obtain

$$\Omega W_{n_{\max}}(\Phi) = \mathrm{e}^{-\Omega V_{\mathrm{eff}}(\Phi/\Omega)}, \tag{3.3}$$

with

$$\Omega = 2^{n_{\max}}. \tag{3.4}$$

Remarkably, the symmetries of the model have allowed us to calculate the partition function and the effective potential by calculating only $(n_{\max})$-independent integrals instead of the $(2^{n_{\max}})$-coupled integrals that one would naively expect.

The symmetry group of the HM is of order $2^{\Omega-1}$, which is one half of the number of configurations for an Ising measure. This can be seen as follows. In any of the blocks of size 2 we can interchange two sites. The exchange can be done independently in each of the blocks and this symmetry can be seen as *local*. There are $\Omega/2$ blocks of size 2 and so $2^{\Omega/2}$ distinct symmetry transformations. Similarly, we can interchange the blocks of size 2 inside any block of size 4 which generates $(2^{\Omega/4})$-independent transformations. We can similarly generate independent symmetries until we reach the group of order 2 exchanging two blocks of size $\Omega/2$. Using

$$1/2 + 1/4 + \cdots + (1/2)^{n_max} = 1 - 1/\Omega,$$

we obtain the announced result.

3.2. The RG transformation

As explained in subsection 2.2, the RG transformation consists in a blockspinning and a rescaling. In the present example, the rescaling is fixed by the requirement of keeping H unchanged in the infinite volume limit. More specifically, if we define

$$\phi'_{x'_{n_{\max}-1},\ldots,x'_1} = \sqrt{\frac{c}{4}}\left(\phi_{x_{n_{\max}},\ldots,x_2,1} + \phi_{x_{n_{\max}},\ldots,x_2,0}\right) \tag{3.5}$$

with

$$x'_{n_{\max}-1} = x_{n_{\max}}, \ldots, x'_1 = x_2, \tag{3.6}$$

we can rewrite the energy as

$$H = H' - \frac{1}{2}\sum_{x'_{n_{\max}-1},\ldots,x'_1}\left(\phi'_{(x'_{n_{\max}-1},\ldots,x'_1)}\right)^2, \tag{3.7}$$

with

$$H' = -\frac{1}{2}\sum_{n=1}^{n_{\max}-1}\left(\frac{c}{4}\right)^n \sum_{x'_{n_{\max}-1},\ldots,x'_{n+1}}\left(\sum_{x'_n,\ldots,x'_1}\phi'_{(x'_{n_{\max}-1},\ldots,x'_1)}\right)^2. \tag{3.8}$$

In other words, we can blockspin without thinking about H and then include the second term of equation (3.7) in the local measures. The problem is then identical to the original problem except for the fact that we have to reduce $n_{\max}$ by 1 and to use a new local measure.

The change in the local measure can be expressed through the recursion relation

$$\mathcal{W}_{n+1}(\phi') = N_{n+1}\exp((\beta/2)\phi'^2)\int_{-\infty}^{+\infty} d\xi\, \mathcal{W}_n\left(\frac{\phi'}{\sqrt{c}} + \xi\right)\mathcal{W}_n\left(\frac{\phi'}{\sqrt{c}} - \xi\right), \tag{3.9}$$

where N_{n+1} is a normalization factor which can be fixed at our convenience. The symbol $\mathcal{W}$ has been used instead of W in order to specify that the field had been rescaled at every step. The explicit relationship is

$$\mathcal{W}_n((c/4)^{n/2}\phi_n) = W_n(\phi_n), \tag{3.10}$$

where ϕ_n denotes the sum of all the fields in a block of size 2^n. Introducing the Fourier representation

$$\mathcal{W}_n(\phi) = \int_{-\infty}^{+\infty}\frac{dk}{2\pi}\, e^{ik\phi} R_n(k), \tag{3.11}$$

the recursion formula becomes

$$R_{n+1}(k) = C_{n+1}\, e^{-\frac{1}{2}\beta\frac{\partial^2}{\partial k^2}}(R_n(\sqrt{c/4}k))^2, \tag{3.12}$$

with C_{n+1} being another arbitrary normalization constant related to the previous one by the relation $C_{n+1} = (c/\sqrt{2})N_{n+1}$. We will fix the normalization constant C_n is such a way that $R_n(0) = 1$. $R_n(k)$ has then a direct probabilistic interpretation. If we call ϕ_n the total unrescaled field $\sum\phi_x$ inside blocks of side 2^n and $\langle\cdots\rangle_n$ the average calculated without taking into account the interactions of level strictly larger than n, we can write

$$R_n(k) = 1 + \sum_{q=1}^{\infty}\frac{(-ik)^q}{q!}(c/4)^{qn/2}\langle(\phi_n)^q\rangle_n. \tag{3.13}$$

We see that the Fourier transform of the local measure after n iterations generates the zero-momentum Green's functions calculated with 2^n sites.

Everything that has been done in this section can be generalized in a straightforward manner for models with N components. All we need to do is to replace ϕ by a N-dimensional vector $\vec{\phi}$, k by $\vec{k}$ and $d\xi$ by $d^N\xi$. The N-component model is discussed in section 13.

3.3. The Gaussian (UV) fixed point

In the case where the initial measure is Gaussian

$$W_0(\phi) = \mathcal{W}_0(\phi) = \mathrm{e}^{-A\phi^2}, \tag{3.14}$$

we have

$$\mathcal{W}_n(\phi) \propto \mathrm{e}^{-A_n\phi^2}, \tag{3.15}$$

with $A_0 = A$ and the A_n calculable from the recursion relation

$$A_{n+1} = -\beta/2 + (2/c)A_n \tag{3.16}$$

which follows from equation (3.9). The fixed point of this transformation is

$$A^\star = \frac{\beta c}{2(2-c)}. \tag{3.17}$$

With our assumption $0 < c < 2$, $A^\star$ is positive and finite. After resumming the terms, we obtain

$$A_n = A^\star + (A - A^\star)(2/c)^n. \tag{3.18}$$

Using equation (3.10), we then find that the measure for the main field Φ is

$$W_{n_{\max}}(\Phi) \propto \mathrm{e}^{-A_{n_{\max}}\Phi^2(c/4)^{n_{\max}}}. \tag{3.19}$$

This implies that

$$V_{\mathrm{eff}}(\phi_c) = ((A - A^\star) + A^\star(c/2)^{n_{\max}})\phi_c^2. \tag{3.20}$$

Given that $0 < c < 2$, the second term disappears in the infinite volume limit and if $A > A^\star$, we can interpret $A - A^\star$ as a quantity proportional to the square of the mass. On the other hand, if $A = A^\star$, we have a massless theory in the infinite volume limit. At finite volume, we expect that for $A = A^\star$,

$$m^2 \propto (c/2)^{n_{\max}} \propto L^{-2}, \tag{3.21}$$

with L being the linear size of the whole system defined by

$$L^D = \Omega = 2^{n_{\max}}. \tag{3.22}$$

Putting these two equations together, we obtain

$$c = 2^{1-2/D}. \tag{3.23}$$

In other words, the parameter c can be tuned in such a way that a Gaussian massless field scales with the number of sites in the same way as a D-dimensional model.

Following the same reasoning, we can determine the parameters introduced in subsection 2.2. Since we integrate over two sites in one RG transformation, we have

$$\ell^D = 2. \tag{3.24}$$

On the other hand, comparing the field rescaling, we obtain

$$(4/c) = \ell^{+D+2-\eta}, \tag{3.25}$$

which is consistent with equation (3.23) only if

$$\eta = 0. \tag{3.26}$$

In summary, we have

$$c = \ell^{D-2}. \tag{3.27}$$

We can reinterpret equation (3.18) by noting that it implies that at each RG transformation, $A_n - A^\star$ is rescaled by a factor $2/c$. If we introduce some reduced variable $\rho_n \equiv (A_n - A^\star)/A^\star$, then

$$\rho_{n+1} = (2/c)\rho_n = \ell^2 \rho_n. \tag{3.28}$$

With the notations introduced in equation (2.28), we could say that the exponent $\mathcal{Y}$ at the Gaussian fixed point associated with the mass term squared is 2.

It should also be said that the Gaussian fixed point is often called the UV fixed point because, it is common to use perturbation theory about the Gaussian fixed point to construct a RG flow from the Gaussian fixed point to the nontrivial fixed point (which is then called an IR fixed point).

3.4. The HT fixed point

In the high-temperature (HT) limit, $\beta = 0$, equation (3.12) becomes

$$R_{n+1}(k) = (R_n(\sqrt{c/4}k))^2. \tag{3.29}$$

We set $C_{n+1} = 1$ and require $R(0) = 1$. It is easy to check that

$$R(k) = \mathrm{e}^{Bk^{2\ln 2/\ln(4/c)}} = \mathrm{e}^{Bk^{2D/(D+2)}} \tag{3.30}$$

is a fixed point of equation (3.29) for arbitrary B. However, if $B \neq 0$ and the exponent of k is not a positive integer, we have a branch cut at $k = 0$ and by taking sufficiently many derivatives with respect to k, we obtain expressions which blow up at $k = 0$. For the range of values $0 < c < 2$, the only way to get a power of k in the exponential which is an integer is to have $c = 1$, which according to equation (3.23) corresponds to $D = 2$. For $c = 1$, the exponent of k is 1, and the Fourier transform in equation (3.11) is ill defined unless we replace k by $|k|$ which in turn leads to singular derivatives at $k = 0$. Consequently, the only choice that leads to a probability distribution with finite moments is $B = 0$, or in other words, $R = 1$. This fixed point remains a fixed point when $\beta \neq 0$ and is called the HT fixed point hereafter. The HT fixed point corresponds to an arbitrarily narrow probability distribution about 0 for the main field and can be interpreted as the case of an arbitrarily massive free field.

4. Equivalent forms of the recursion formulae

The recursion formula will be the main tool used for calculations hereafter. It is important to identify equivalent or inequivalent forms and to find accurate numerical implementations. In this section, we review equivalent forms of the recursion formulae used by Baker [6], Felder [13] and Koch and Wittwer [49, 50]. This section follows closely [50] but with different notations. In the following, the constants $N'_n, N''_n, \ldots$ need to be fixed by some additional requirement and play no essential role.

4.1. Baker's form

It is sometimes convenient to factor out the Gaussian fixed point so that in a new 'system of coordinates', the Gaussian fixed point is represented by a constant. If we define

$$\mathcal{W}_n(\phi) = \exp(-A^\star \phi^2)\mathcal{H}_n(\phi), \tag{4.1}$$

the recursion formula becomes

$$\mathcal{H}_{n+1}(\phi) = N'_{n+1} \int_{-\infty}^{+\infty} \mathrm{d}\xi \exp(-2A^\star \xi^2)\mathcal{H}_n\left(\frac{\phi}{\sqrt{c}} + \xi\right)\mathcal{H}_n\left(\frac{\phi}{\sqrt{c}} - \xi\right). \tag{4.2}$$

It is clear from this equation that if $\mathcal{H}_n$ is a constant then $\mathcal{H}_{n+1}$ is also constant. What is remarkable about this redefinition is that we have replaced $\exp((\beta/2)\phi^2)\,\mathrm{d}\xi$ in equation (3.9) by $\mathrm{d}\xi\exp(-2A^\star\xi^2)$. Any other multiplicative redefinition would, in general, lead to some hybrid form where the weight depends on ϕ and ξ.

It should also be noted that the variance of the Gaussian weight can be replaced by another value by a simple change of variable. If we define

$$\mathcal{H}^{[B]}(\phi) \equiv \mathcal{H}(B\phi), \tag{4.3}$$

the recursion formula becomes

$$\mathcal{H}^{[B]}_{n+1}(\phi) = N'_{n+1}B\int_{-\infty}^{+\infty}\mathrm{d}\xi\,\exp(-2A^\star B^2\xi^2)\mathcal{H}^{[B]}_n\left(\frac{\phi}{\sqrt{c}}+\xi\right)\mathcal{H}^{[B]}_n\left(\frac{\phi}{\sqrt{c}}-\xi\right). \tag{4.4}$$

The recursion used by Baker in [6] is obtained by setting

$$\mathcal{H}^{[(K/2A^\star)^{1/2}]}_n(\phi) = \exp\bigl(-\tfrac{1}{2}Q_n(\phi)\bigr). \tag{4.5}$$

Note that Baker allowed a non-zero η in order to mimic the scaling near the nontrivial fixed point instead of the scaling near the Gaussian fixed point for a conventional lattice model. It is possible to take into account this modification by changing the value of D used in equation (3.23). Confusion can be avoided by making reference to the rescaling factor $c^{-1/2}$. More explicitly, if we call D' the dimension used for a field rescaling that depends on η, then the correspondence is

$$2^{(2-\eta-D')/2D'} = c^{-1/2} = 2^{(2-D)/2D}. \tag{4.6}$$

A short calculation shows that $D' = D(1-(\eta/2))$. As an example, if we want to have a scaling corresponding to $D' = 3$ and $\eta = 0.04$, we can simply work with $D = 3.061,\ldots,$ with D defined in equation (3.23). Using notation that we hope making clear the relation with [6], the recursion formula can then be written as

$$\exp\left(-\frac{1}{2}Q_{n+1}(\phi)\right) = N''_{n+1}\int_{-\infty}^{+\infty}\mathrm{d}\xi\,\exp\left(-K\xi^2-\frac{1}{2}Q_n\left(\frac{\phi}{\sqrt{c}}+\xi\right)-\frac{1}{2}Q_n\left(\frac{\phi}{\sqrt{c}}-\xi\right)\right). \tag{4.7}$$

4.2. Gallavotti's form

An alternate way of formulating the recursion is to use a convolution with the Gaussian fixed point

$$\mathcal{G}_n(t) = \int_{-\infty}^{+\infty}\mathrm{d}\phi\,\exp(-A^\star(t-\phi)^2)\mathcal{H}_n(\phi). \tag{4.8}$$

By factoring out the two exponentials of the quadratic terms, it is possible to relate $\mathcal{G}$ to the Fourier transform of $\mathcal{W}$ introduced in equation (3.12). More explicitly

$$\exp(A^\star t^2)\mathcal{G}_n(t) \propto R_n(i2tA^\star), \tag{4.9}$$

and in particular the HT fixed point is now proportional to $\exp(-A^\star t^2)$. Under the new transformation, the recursion formula takes the form

$$\mathcal{G}_{n+1}(t) = N'''_{n+1}\int_{-\infty}^{+\infty}\mathrm{d}\xi\,\exp\left(-2\frac{A^\star c}{2-c}\xi^2\right)\left(\mathcal{G}_n\left(\frac{t}{\sqrt{c}}+\xi\right)\right)^2. \tag{4.10}$$

This form of the recursion formula was used in [13, 49, 50], and its origin can be found in the work of Gallavotti [51]. More about this question can be found in subsection 5.2 below. In the following, we call this form of the recursion formula Gallavotti's form.

Table 1. Form of the fixed point up to an overall constant in the various coordinates. 1 is short for a constant and δ is short for a delta function; the last line is the defining equation.

	$\mathcal{W}$	$\mathcal{H}$	$\mathcal{G}$	$\mathcal{F}$	R	f
UV	$e^{-A^\star\phi^2}$	1	1	$e^{A^\star t^2}$	$e^{-\frac{k^2}{4A^\star}}$	$e^{\frac{2-c}{4-c}t^2}$
HT	δ	δ	$e^{-A^\star t^2}$	1	1	1
Equation	3.9	4.1	4.8	4.11	3.11	4.15

If we now define

$$\mathcal{G}_n(t) = \exp(-A^\star t^2)\mathcal{F}_n(t), \tag{4.11}$$

the recursion formula becomes after some algebra

$$\mathcal{F}_{n+1}(t) = N'''_{n+1}\int_{-\infty}^{+\infty} d\xi \exp\left(-4\frac{A^\star}{2-c}(\xi - t\sqrt{c/4})^2\right)(\mathcal{F}_n(\xi))^2\,. \tag{4.12}$$

From the definitions of $\mathcal{G}_n$ and $\mathcal{F}_n$, we have

$$\mathcal{F}_n(t) \propto R_n(i2tA^\star), \tag{4.13}$$

and we can prove the equivalence of (4.12) and (3.12) by using the identity

$$\exp\left(-\frac{1}{2}\beta\frac{\partial^2}{\partial k^2}\right)\left(k\sqrt{\frac{c}{4}}\right)^q = \sqrt{\frac{4A^\star}{\pi(2-c)}} \times \int_{-\infty}^{+\infty} d\xi \exp\left(-4\frac{A^\star}{2-c}\left(\xi + i\frac{k}{2A^\star}\sqrt{c/4}\right)^2\right)(2i\xi A^\star)^q. \tag{4.14}$$

Finally, if we write

$$f_n(t) = \mathcal{F}_n\left(t\sqrt{\frac{2-c}{A^\star(4-c)}}\right), \tag{4.15}$$

we obtain the form most often used in [50]:

$$f_{n+1}(t) = N''''_{n+1}\int_{-\infty}^{+\infty} ds \exp\left(-\frac{1}{1-c/4}s^2\right)(f_n(s + t\sqrt{c/4}))^2. \tag{4.16}$$

4.3. Summary

The form of the UV and HT fixed points in the various sets of coordinates is summarized in table 1.

All this looks quite reminiscent of quantum mechanics where we can look at a problem in a basis where the position operator is diagonal or in another basis where the momentum operator is diagonal. The two basis are related by a unitary transformation. More generally, in quantum mechanics, unitary transformations do not affect the spectrum of Hermitian operators (which represent observables). Universality is a stronger notion than observability in the sense that some non-universal quantities may have an absolute physical meaning independent of our choice of integration variables in the functional integral. The idea of transformations that leave universal properties unchanged have been discussed in [52] where this translate into a reparametrization invariance.

5. Inequivalent extensions of the recursion formula

In this section, we discuss more general recursion formulae which coincide with the HM's one for a particular choice of the parameter ℓ that controls the change in the linear scale after one RG transformation.

5.1. Relation with Wilson's approximate recursion formula

Wilson's approximate recursion formula was the first simplified RG transformation that was proposed. It appears in one of the basic RG papers [3] as a result of a rather involved analysis of the partition function of a scalar model with an UV cutoff. It played an important role, because it was realized that the RG method could lead to a fully numerical treatment without any reference to expansions such as perturbation in a weak coupling. It is intended to represent a situation where $\ell = 2$, but instead of having $2^D - 1$ integration variables, it has only one corresponding to an approximation where the 2^D fields take only two independent values [32], one in each half of the block. Having decoupled the number of integration variables to ℓ, we can now write for arbitrary ℓ:

$$H_{n+1}^{[\ell]}(\phi) = N_{n+1}'''''' \int_{-\infty}^{+\infty} \mathrm{d}\xi \, \mathrm{e}^{-\xi^2} \left[H_n^{[\ell]}\left(\ell^{1-\frac{D}{2}}\phi + \xi\right) H_n^{[\ell]}\left(\ell^{1-\frac{D}{2}}\phi - \xi\right) \right]^{\ell^D/2}. \tag{5.1}$$

For $\ell = 2$, we obtain Wilson's approximate recursion formula, most often written in a form similar to equation (4.7). For $\ell = 2^{1/D}$, we obtain the HM recursion formula in the form given in equation (4.2).

The general case $\ell = 2^\zeta$ was discussed in [53] where it was shown that for $D = 3$, as ζ increases from $\zeta = 1/3$ to 1, the exponent γ decreases monotonically from 1.30 to 1.22. Clearly, different values of ζ correspond to different classes of universality.

5.2. Gallavotti's recursion formula

The recursion formulae presented in subsection 4.2 can also be extended for arbitrary ℓ. This is indeed easier because the number of sites integrated for the HM, namely 2, appears as the exponent for $\mathcal{F}$, $\mathcal{G}$ and f. The replacements are

$$2 \to \ell^D \tag{5.2}$$

$$\frac{1}{\sqrt{c}} \to \ell^{1-D/2} \tag{5.3}$$

$$\frac{c}{4} \to \ell^{-2-D}. \tag{5.4}$$

For instance equation (4.16) becomes

$$f_{n+1}^{[\ell]}(t) = N_{n+1}'''''' \int_{-\infty}^{+\infty} \mathrm{d}s \exp\left(-\frac{1}{1-\ell^{-D-2}} s^2\right) \left[f_n^{[\ell]}(s + t\ell^{-D/2-1}) \right]^{\ell^D}. \tag{5.5}$$

Again for $\ell = 2^{1/D}$, we recover the HM recursion formula in the form given in equation (4.16). In [51], Gallavotti has introduced equations that can be identified with the case $\ell = 2$. The variable s in equation (5.5) plays a role similar to the Gaussian variable z_Δ in his notations, and the factor $\ell^D = 2^D$ is reabsorbed in the definition of the potential. The limit $\ell \to 1$ of equation (4.10) will be discussed in section 9.

For $\ell = 2$ and $D = 3$, the value $\gamma = 1.300\,33\ldots$ was obtained numerically in [54]. This value is significantly different from the value $\gamma = 1.299\,1407\ldots$ obtained in [55, 56]

for $\ell = 2^{1/3}$ and $D = 3$. For $D = 3$, the limit $\ell \to 1$ was studied in [17, 57] with the result $\gamma = 1.299\,124$. The difference in the fifth digit is significant and was confirmed by new calculations [19] (see also our section 9 below). This shows that again different values of ℓ correspond to different classes of universality and also that for a given $\ell \neq 2^{1/D}$, the extensions given in equations (5.1) and (5.5) are inequivalent. In addition, we see that the ℓ dependence is much weaker in equation (5.5) and the slope is apparently opposite to the slope found for equation (5.1). We are lacking calculations at intermediate values but as far as we can see, when ℓ increases, γ increases.

6. Motivations, rigourous and numerical results

6.1. Motivations

Dyson's hierarchical model was invented and reinvented several times with different motivations that we briefly review. Dyson's original motivation [5] was to construct models more weakly coupled than the one-dimensional Ising models with long-range Hamiltonians of the form

$$H = -J \sum_{m<n} |n-m|^{-\alpha} \phi_n \phi_m. \tag{6.1}$$

Dyson's was trying to figure out if the model has an ordered phase when $\alpha = 2$. Dyson constructed a more general family of models where c^l is replaced by b_l in equation (3.1). He proved several theorems concerning the infinite volume limit and the existence of phase transitions at finite temperature that are discussed in the following subsection.

The fact that $\alpha = 2$ is borderline can be anticipated by refining [58] Landau's argument for the absence of ordered phase in one dimension. If $\alpha < 2$ and a system of size L has an average magnetization $\mu > 0$, then the cost in energy for flipping all the spins in a large subsystem is of the order of $L^{2-\alpha}$. If $\alpha > 2$, the cost should grow slower than $\ln(L)$ when L is increased. On the other hand, there is of the order of L ways to choose the subsystem and so the gain in entropy is of order $kT \ln L$. Consequently if $\alpha > 2$, the entropy dominates and the free energy is decreased after flipping, which is incompatible with the possibility of an equilibrium situation. On the other hand, if $\alpha < 2$, there is no incompatibility. In the case $\alpha = 2$, both contributions are logarithmic and a more careful estimate is required. Thouless [58] estimated that if the width of the magnetization distribution is proportional to $L^{1/2}$, the change in free energy is

$$\Delta E = 2\mu^2 J \ln L - (1/2)kT \ln L, \tag{6.2}$$

and an ordered state seems possible for T small enough. If this occurs at some strictly positive critical temperature T_c, then the magnetization changes abruptly to $(kT_c/4J)^{1/2}$ when the temperature is lowered to T_c. This is called the Thouless effect. Later, the existence of an ordered state at sufficiently low temperature for the model defined by equation (6.1) with $\alpha = 2$ was proved rigorously [59] as well as the Thouless effect [60].

Baker rediscovered the HM [6] in the context of the development of the RG ideas. His goal was to construct models for which a simple recursion formula would be exact. He reinvented Dyson's HM and several variant of it. This has been partially reviewed in sections 4 and 5.

The hierarchical structure of the block variables can be naturally reconstructed using the 2-adic numbers (see section 14). At the end of the 1980s, physicists started reformulating models of classical, quantum and statistical mechanics over the fields of p-adic numbers [61]. In particular, models of random walks over the p-adic numbers were considered [62–64], and it was recognized that it was possible to reformulate the HM as a scalar model on the 2-adic

fractions [65]. This reformulation was used in high-temperature (HT) expansions [66], helps understanding the absence of certain Feynman graphs [3] and suggests ways to improve the hierarchical approximation [21]. This is discussed in more detail in section 14.

6.2. *Rigourous results*

Dyson proved several theorems for the HM with

$$c = 2^{2-\alpha}, \tag{6.3}$$

where α is the same as in equation (6.1). With this notation it is clear that the ferromagnetic interactions are weaker than for the model of equation (6.1). If we refer to figure 1, the relative strength of the couplings between the 0th spin and its right neighbours are, from left to right, $1, 2^{-\alpha}, 3^{-\alpha}, \ldots, 7^{-\alpha}$ for equation (6.1) and $2^{-\alpha}, 4^{-\alpha}, 4^{-\alpha}, 8^{-\alpha}, 8^{-\alpha}, 8^{-\alpha}, 8^{-\alpha}$ for the HM. Griffiths has proved that for Ising models with ferromagnetic interactions, the averages of two arbitrary spins variables are positive and increase with the strength of the ferromagnetic interactions. Consequently, if one can prove that $\langle \phi_n \sum_m \phi_m \rangle$ blows up in the infinite volume limit below some temperature for the more weakly coupled model, it will also blow up for the other model. Dyson proved the existence of the infinite volume limit for $\alpha > 1$ and that there was a phase transition at finite temperature if and only if $1 < \alpha < 2$. In other words, there is no phase transition at $\alpha = 2$ for the HM, but this does not allow us to extend the result to the more strongly coupled model of equation (6.1) that has indeed a phase transition at finite temperature [59]. With the notations used in section 3, Dyson theorems mean that an infinite volume limit exists for $c < 2$ ($D > 0$) and that a phase transition at finite temperature occurs if $1 < c < 2$ ($D > 2$).

The HM has been studied near $D = 4$ ($c = \sqrt{2}$) using the ϵ expansion. The existence of a nontrivial fixed point for ϵ small enough was proved [67]. The ϵ expansion was shown to be asymptotic [68]. Many details regarding this approach can be found in [30]. Extensions beyond the hierarchical model are discussed in [69].

The HM has also been studied directly at $D = 3$ ($c = 2^{1/3}$). The existence of a nontrivial fixed point was proved for a large enough number of components [70]. Proofs of the existence of the nontrivial fixed point in $D = 3$ for the one component model were given in [49, 50, 54, 71, 72]. They also put exponential bounds on the fixed point in various representations discussed in section 4. In particular, for real ϕ, some positive C and $\mathcal{H}^\star$ the nontrivial fixed point of equation (4.2), the following bound holds:

$$|\mathcal{H}^\star(\phi)| < \exp(-C\phi^6). \tag{6.4}$$

6.3. *Numerical results*

The critical and tricritical behaviour of the HM was investigated numerically in the presence of a magnetic field and a staggered magnetic field [73]. The literature contains many numerical estimates of the critical exponent γ for $D = 3$. The first calculation was done by Wilson. The result reported in [6] is $\gamma = 1.2991$. In the following, unless specified differently, errors of order 1 should be assumed for the last printed digit. Values when $2/D$ is a multiple of 0.05 are given in [74]. Interpolating linearly to $D = 3$, we obtained 1.302. Using the ϵ expansion up to order 34 and a Borel resummation method, the value 1.2986 was obtained in [75]. Analysis of the HT expansion [27, 28] yields 1.300(2) for $D = 3$. The value 1.299 141 can be obtained from a footnote in [54]. The value 1.299 14 was obtained in [76]. Using two independent methods discussed in section 7, the value 1.299 140 730 159 was found in [55, 56]. Less accurate calculations in the low-temperature phase were performed in [77] and confirmed hyperscaling with three decimal points.

7. Numerical implementation

7.1. Polynomial truncations

The recursion formula can be implemented numerically using numerical integration methods in equations (4.2) or (5.1) as was done for instance in [3, 53, 73]. However, in the symmetric phase, it seems easier to get very accurate results by using polynomial approximations in forms based on the Fourier transform such as equations (3.12) or (4.16). This method was justified rigorously in [54, 72], and used for numerical calculations for instance in [14, 56, 76, 78]. For definiteness, we start with the recursion formula for R given in equation (3.12). A finite-dimensional approximations of degree $l_{\max}$ has the form

$$R_n(k) = 1 + a_{n,1}k^2 + a_{n,2}k^4 + \cdots + a_{n,l_{\max}}k^{2l_{\max}}. \tag{7.1}$$

This type of approximation can be justified in the context of the HT expansion and works extremely well in the symmetric phase. We can reabsorb the inverse temperature β in k in such a way that it does not appear anymore in the exponential in equation (3.12). This change would then be compensated by a transformation $a_{n,l} \to \beta^l a_{n,l}$ and the truncation at order $k^{2l_{\max}}$ would be sufficient to calculate exactly the HT expansion of R up to order $\beta^{l_{\max}}$. This technique was used in [27, 28]. It was then realized (by accident) that large-order coefficients in the HT expansion could be calculated in good approximation by using polynomial truncations at order 10 times smaller than the HT order. The same method can be applied to numerical calculations in the symmetric phase. The apparent convergence is studied empirically in [78].

With the polynomial truncation, the recursion formula equation (3.12) becomes a $l_{\max}$-dimensional quadratic map. After squaring R, we obtain a polynomial of order $2l_{\max}$ in k^2. We could in principle truncate at order $l_{\max}$; however, the derivatives in the exponential in equation (3.12) will lower the degree and the terms of order larger than $l_{\max}$ will contribute to the orders smaller than $l_{\max}$ after enough derivatives are applied. Of course, a truncation at order $l_{\max}$ is made after all the derivatives are performed, but it was realized empirically [79] that intermediate truncations reduce the accuracy of the calculation. The explicit algebraic transformation reads

$$a_{n+1,m} = \frac{\sum_{l=m}^{2l_{\max}} \left(\sum_{p+q=l} a_{n,p}a_{n,q}\right)[(2l)!/(l-m)!(2m)!](c/4)^l[-(1/2)\beta]^{l-m}}{\sum_{l=0}^{2l_{\max}} \left(\sum_{p+q=l} a_{n,p}a_{n,q}\right)[(2l)!/l!](c/4)^l[-(1/2)\beta]^l}. \tag{7.2}$$

The initial conditions for the Ising measure is $R_0(k) = \cos(k)$. For the LG measure, the coefficients in the k expansion need to be evaluated numerically. The susceptibility at finite volume and higher moments can then be obtained by rescaling the coefficients, for instance,

$$\chi_n^{(2)} = -2a_{n,1}(2/c)^n. \tag{7.3}$$

7.2. Volume effects in the symmetric phase

When calculating the susceptibility at values of β slightly below β_c, we spend about $-\ln(\beta_c - \beta)/\ln(\lambda_1)$ iterations near the fixed point. During these iterations, the round-off errors are amplified along the unstable direction (see the following subsection). After that the order of magnitude of the susceptibility stabilizes, and the corrections get smaller by a factor $\frac{c}{2}$ at each iterations. At some point, all the recorded digits stabilize (irrespectively of the numerical errors which occurred in the first stage described above). This gives the estimate [78] for the number of iterations $n(\beta, P)$ to stabilize P digits (in decimal notations)

$$n(\beta, P) = \left(\frac{D\ln(10)}{2\ln(2)}\right)[P - \gamma \log_{10}(\beta_c - \beta)]. \tag{7.4}$$

7.3. The eigenvalues of the linearized RG transformation

The critical exponents can be calculated by linearizing the RG transformation near the fixed point $R^\star(k)$ specified by the coefficients $a^\star{}_l$. We express the coefficients after n iterations in terms of small variations about the fixed point

$$a_{n,l} = a_l^\star + \delta a_{n,l}. \tag{7.5}$$

At the next iteration, we obtain the linear variations

$$\delta a_{n+1,l} = \sum_{m=1}^{l_{\max}} M_{l,m} \delta a_{n,m}. \tag{7.6}$$

The $l_{\max} \times l_{\max}$ matrix appearing in this equation is

$$M_{l,m} = \frac{\partial a_{n+1,l}}{\partial a_{n,m}}, \tag{7.7}$$

evaluated at the fixed point.

Approximate fixed points can be found by approaching β_c from below and iterating until the ratio $a_{n+1,1}/a_{n,1}$ takes a value which is as close as possible to 1. The determination of β_c can be done by following the bifurcations in $a_{n+1,1}/a_{n,1}$ for sufficiently large n. When $\beta < \beta_c$, the susceptibility stabilizes at a finite value without subtraction and for n large enough, $a_{n+1,1}/a_{n,1} \to c/2$. On the other hand, if $\beta > \beta_c$, the unsubtracted susceptibility grows like the volume and for n large enough, $a_{n+1,1}/a_{n,1} \to c$ (until the polynomial truncation breaks down).

The approximated fixed points obtained with this procedure depend on β_c. Using their explicit form which we denote by $R^\star(k, \beta_c)$, one obtains a universal function $\mathfrak{R}(k)$ by absorbing β into k:

$$\mathfrak{R}(k) = R^\star(\sqrt{\beta_c}k, \beta_c). \tag{7.8}$$

It was shown that in very good approximation, $\mathfrak{R}(k)$ is independent of the initial measure considered [55, 56]. Numerically,

$$\mathfrak{R}(k) = 1. - 0.358\,711\,349\,88k^2 + 0.053\,537\,2882k^4 - \cdots. \tag{7.9}$$

This function is related to the fixed point $f_{KW}(s^2)$ constructed in [72] as follows by the relation

$$\mathfrak{R}(k) \propto f_{KW}\left(\left(\frac{c-4}{2c}\right)k^2\right). \tag{7.10}$$

Extremely accurate values of the Taylor coefficients of f_{KW} can be found in the file `approx.t` in [72]. The constant of proportionality is fixed by the condition $\mathfrak{R}(0) = 1$. The relation with the nontrivial fixed point $f^\star$ of equation (4.16) is $f^\star(it) = f_{KW}(-t^2)$.

The first six eigenvalues of $M_{l,m}$ from [56] are given in table 2.

Using $\gamma = \ln(2/c)/\ln\lambda_1$ and $\Delta_s = -\ln\lambda_2/\ln\lambda_1$ from section 2, we obtain

$$\gamma = 1.299\,140\,730\,159 \tag{7.11}$$

$$\Delta_s = 0.425\,946\,858\,988. \tag{7.12}$$

These estimates of the exponents were in agreement [56] with those obtained from fits of numerical data near criticality based on the parametrization of equation (2.26) with 12 significant digits for γ and 6 significant digits for Δ_s. These fits also provide the non-universal amplitudes.

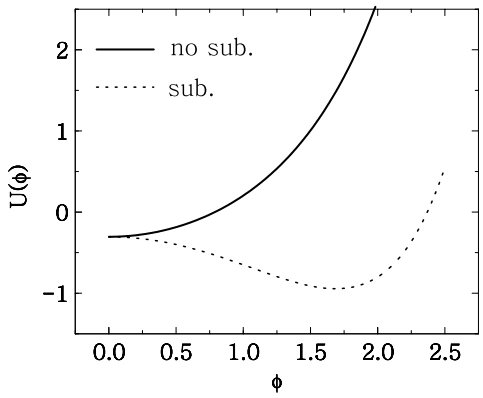

Figure 2. $U(\phi)$ defined in equation (7.13) (solid line); the subtracted potential $U(\phi) - (c/(2(2-c)))\phi^2$ (dotted line).

Table 2. The fist six eigenvalues of the linearized RG transformation

n	λ_n
1	1.427 172 478 177 59
2	0.859 411 649 182 006
3	0.479 637 305 387 532
4	0.255 127 961 414 034
5	0.131 035 246 260 843
6	0.065 488 493 129 8533

7.4. The critical potential

It is possible to Fourier transform numerically $\mathfrak{R}(k)$ for a not too large value of the conjugate variable ϕ. The result is that apparently the Fourier transform is a positive bell-shaped function with only one maximum at $\phi = 0$. We define

$$U(\phi) \equiv -\ln \widehat{\mathfrak{R}}(\phi), \tag{7.13}$$

and we obtain an apparently convex function with only one minimum shown as the solid line in figure 2. In order to compare with calculations done with the $\mathcal{H}_n$ formulation, we need to subtract $A^\star\phi^2$ from this function. Since we have removed the β dependence by reabsorbing it in the definition of k, we must calculate $A^\star$ with $\beta = 1$. Subtracting $(c/(2(2-c)))\phi^2 \simeq 0.851\phi^2$ from the rescaled potential then becomes a double-well potential with minima at $\phi \simeq \pm 1.688$. This is illustrated in figure 2. This explains that figures with a minimum away from the origin appear in [32, 73].

It should be emphasized that $U(\phi)$ is a rescaled potential and not the effective potential. The relationship between the two can be worked out from equations (3.10) and (3.11). Assuming we are exactly at the nontrivial fixed point, $\mathcal{W}$ remains $\widehat{\mathfrak{R}}(\phi)$. One then obtains

$$Z \propto R^\star(-iJ(4/c)^{n_{\max}/2}), \tag{7.14}$$

and

$$\Omega V_{\text{eff}}(\phi_c) = U(\phi_c c^{n_{\max}/2}). \tag{7.15}$$

This result can be interpreted in the following way: at the fixed point, the only scale in the problem is the size of the system and all the quantities scale with it according to their

dimension. If we perform Taylor expansions in equations (7.14) and (7.15), we see that when $n_{\max} \to \infty$, all the coefficients of Z blow up and the coefficients of ϕ_c^{2r} blow up if $r > D/(D-2)$. In the same limit, the coefficients of ϕ_c^{2r} tend to zero if $r < D/(D-2)$. The limiting case corresponds to $r_c = D/(D-2)$ where the coefficient is volume independent. For instance for $D = 4$, $r_c = 2$ and for $D = 3$, $r_c = 3$ which corresponds in both cases to the marginal direction in perturbation theory. It should however be noted that if we work at finite ϕ_c, we are in fact probing U at large value of its argument and the Taylor expansion may not be convergent (the radius of convergence should be determined by the complex zero of $\widetilde{\mathfrak{R}}(\phi)$ closest to the origin). The above determination of r_c implies that

$$U(\phi) \propto |\phi|^{2D/(D-2)} = |\phi|^{D/x_h} \qquad \text{for} \quad |\phi| \to \infty. \tag{7.16}$$

For $D = 3$, the ϕ^6 behaviour is compatible with the rigourous bound of equation (6.4).

Equation (7.16) can be obtained from the scaling hypothesis discussed at the end of section 2.2. First, we find from equation (7.14) that

$$\ell^D G[J] = -\ln(R^\star(-i J \ell^{(D+2)/2})) + \text{constant}, \tag{7.17}$$

which implies that for large values of $|J|$ and $|\phi|$,

$$G[J] \propto |J|^{2D/(D+2)} = |J|^{D/y_h}, \tag{7.18}$$

and

$$|\phi| \propto |J|^{(D-2)/(D+2)}. \tag{7.19}$$

Combining the two above equations and the Legendre transform, we recover equation (7.16).

7.5. The low-temperature phase

The calculation in the low-temperature phase requires the introduction of a constant external magnetic field coupling linearly to ϕ_n. Since

$$W_n(\phi_n, H) \propto W_n(\phi_n)\, \mathrm{e}^{H\phi_n}, \tag{7.20}$$

after Fourier transforming and rescaling one obtains [77]

$$\frac{R_n(k + iH(4/c)^{n/2})}{R_n(iH(4/c)^{n/2})} = \sum_{q=0}^{\infty} \frac{(-ik)^q}{q!} \langle(\phi_n)^q\rangle_{n,H} (c/4)^{qn/2}. \tag{7.21}$$

The connected Green's functions can be obtained by taking the logarithm of this generating function. The average are understood at non-zero H. In order to observe the magnetization, it is essential to take the infinite volume limit *before* taking the limit $H \to 0$. For any non-zero H, no matter how small its absolute value is, one can always find an n large enough to have $|H(4/c)^{n/2}| \gg 1$. The nonlinear effects are then important and linearization does not apply. It was checked [77] that when such an n is reached, the value of the $\chi^{(q)}$ stabilizes at an exponential rate. One can then, *first* extrapolate at infinite volume for a given magnetic field, and *then* reduce the magnetic field in order to extrapolate a sequence of infinite volume limits with a decreasing magnetic field, towards the zero magnetic field.

In the following, we use the notation

$$\chi^{(q)} \propto (\beta - \beta_c)^{-\gamma_q}, \tag{7.22}$$

for the leading exponent. It is customary to use the notation $\gamma_1 = -\beta$, but we avoided it here because of a possible confusion with the inverse temperature. Obviously $\gamma_2 = \gamma$. In the symmetric phase and for q even, we have the order of magnitude estimate

$$\chi^{(q)} \approx 2^{-n^\star} (4/c)^{qn^\star/2}, \tag{7.23}$$

with $n^\star$ defined as in section 2 by the relation $|\beta - \beta_c|\lambda^{n^\star} = 1$. Eliminating $n^\star$ and using the expression of γ as in section 2, we obtain

$$\gamma_q = \gamma[(q/2)\ln(4/c) - \ln 2]/\ln(2/c). \tag{7.24}$$

In the case $D = 3 (c = 2^{1/3})$, this becomes

$$\gamma_q = 1.299\,140\,73\ldots \times (5q - 6)/4. \tag{7.25}$$

We will show below that this equation is a particular case of a more general relation that follows from the scaling hypothesis. In [77], the following numerical results were obtained:

$$\begin{aligned} \gamma_1 &= -0.3247 \\ \gamma_2 &= 1.2997 \\ \gamma_3 &= 2.9237, \end{aligned} \tag{7.26}$$

which agree with three significant digits with the prediction of equation (7.25).

Equation (7.24) follows from a slightly stronger form of the scaling hypothesis. The basic scaling relation of equation (2.27) is satisfied if we further assume that

$$G_s(t, J) = t^{D\nu} g(J/t^{\Delta_g}), \tag{7.27}$$

for a well-behaved function g and with the gap exponent

$$\Delta_g \equiv \frac{\mathcal{Y}_h}{\mathcal{Y}_t} = \frac{(D + 2 - \eta)\nu}{2}. \tag{7.28}$$

By construction, the argument of g is invariant under the rescaling of equation (2.27). Each derivative with respect to J brings down a factor $t^{-\Delta_g}$. This implies that

$$\gamma_q = -D\nu + q\Delta_g. \tag{7.29}$$

In the case of the HM, $\nu = \gamma/2$ and $\Delta_g = (D + 2)\gamma/4$ and we recover equation (7.25) for $D = 3$. For the HM in arbitrary dimension, we could also write

$$\gamma_q = -\frac{D - (q/2)(D + 2)}{\mathcal{Y}_t}. \tag{7.30}$$

7.6. Practical aspects of the hierarchy problem

In the absence of wavefunction renormalization, the square of the renormalized mass m_R in units of the UV cutoff Λ can be defined as the inverse susceptibility. Keeping the mass small when the cutoff increases requires a large susceptibility. In the calculations discussed above, a large susceptibility is obtained by fine-tuning β. However, we can also keep $\beta = 1$ and fine tune another parameter such as the bare mass m_B in a Landau–Ginzburg potential. In this case, we have

$$m_R/\Lambda \sim (|m_{Bc} - m_B|/\Lambda)^{\gamma/2}, \tag{7.31}$$

In four dimensions, $\gamma = 1$ and if we take $m = 100$ GeV, a typical electroweak scale, and $\Lambda = 10^{19}$ GeV of the order of the Planck mass, we need to fine tune m_B with 34 digits. This is often called the hierarchy problem and seen as an argument against fundamental scalars [80]. The main virtue of the RG approach is to separate the relevant and irrelevant parts of the information contained in the partition function. At each iteration, the information relevant to understand the large distance behaviour is amplified, while the rest of the information is discarded according to its degree of irrelevance. However, if some 'noise' is introduced in this process, for instance, as round-off errors in the calculation, the error in the relevant direction will be amplified too. This may lead to situations where the amplified errors wipe out the final result. In the case of the HM, the problem can be solved by increasing the arithmetic precision in the implementation of equation (7.2). This is documented in [81].

8. Perturbation theory with a large field cutoff

8.1. Feynman rules and numerical perturbation theory

An attractive feature of the HM is that it is possible to calculate perturbative series to large order by blockspinning numerically, order by order in an expansion parameter, for instance λ for a $\lambda\phi^4$ perturbation. This method can be used analytically to reconstruct the Feynman rules [16, 82]. In practice, the diagrammatic expansion is much more complicated than the numerical method. However, for comparison with calculations based on diagrams, it is useful to know the Feynman rules.

For an initial measure of the form

$$W_0(\phi) = \exp\bigl(-\bigl(A^\star + \tfrac{1}{2}m_B^2\bigr)\phi^2 - \lambda\phi^4\bigr), \tag{8.1}$$

we obtain the usual Feynman rules for a $\lambda\phi^4$ theory with the following replacements:

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \to \sum_{n=0}^{\infty} 2^{-n-1} \qquad \frac{1}{k^2 + m_B^2} \to \frac{1}{k^2(n) + m_B^2}, \tag{8.2}$$

with $k^2(n) = 2A^\star(c/2)^n$. The interpretation is quite simple; the integral over the momenta is replaced by a sum over momentum shells similar to those introduced by Wilson [3, 32]. After one RG transformation the UV cutoff Λ is lowered to $\Lambda/\ell = 2^{-1/D}\Lambda$, and the volume of momentum space in D dimension is reduced by a factor 2. The volume of the 0th shell is $1/2$, the volume of the 1st shell is $1/4$ etc.... Similarly, $(c/2)^n = \ell^{-2n}$ represents the square of the momentum in the nth shell.

8.2. Perturbation theory with a large field cutoff

It is well known [83] that perturbative series are in general divergent. Their zero radius of convergence is due to large field configurations [84, 85]. However, the large field configurations have very little contributions to observables involving a few fields such as the magnetic susceptibility or the four-point function.

This point was realized by the author in two different circumstances. The first is quantum mechanics, quantum field theory in 1+0 dimensions, where the field variable is usually denoted by x. A large field cutoff can be implemented by imposing that the potential becomes $+\infty$ at $x = \pm x_{\max}$. If the field cutoff $x_{\max}$ is large enough, the effects on the low energy levels are exponentially small [86–88]. The second circumstance is the HM [78]. The numerical procedure described in section 7 is based on polynomial approximations and is purely algebraic; however, we need to input $R_0(k)$. So we need to do one integral numerically to start, namely the inverse Fourier transform of equation (3.11). At the end, we need to Fourier transform if we want to extract the effective potential. In doing the initial integral numerically it is convenient to introduce a large field cutoff and then monitor the effect of this cutoff when it is increased. It is clear that for local measures that decay sufficiently fast, the effect is exponentially small for observables involving a few fields.

On the other hand, the large order of the perturbative series involves averages of large powers of the field and is sensitive to the field cutoff. This is illustrated in figure 3 where the perturbative coefficients of the zero-momentum two-point function for $D = 3$ are plotted in units of their value at infinite field cutoff as a function of the field cutoff. One can see that for a fixed large field cutoff $\phi_{\max}$, some low-order coefficients may be close to their asymptotic values, a few coefficient may be in the crossover region and most coefficients are much smaller by several order of magnitude than their asymptotic value. These three regimes are reminiscent of the three regimes encountered when calculating renormalization group flows between two

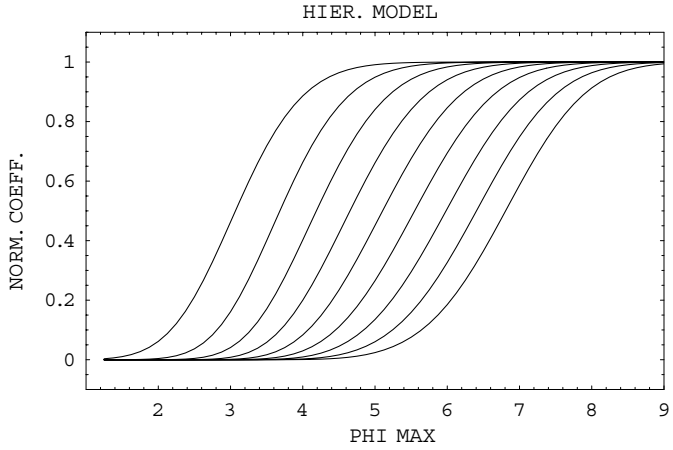

Figure 3. First nine perturbative coefficients (left to right) for the two-point function in unit of their infinite field cutoff value, as a function of the field cutoff $\phi_{\max}$.

fixed points [38, 79, 89, 90]. Note also that the shape of the transition seems universal as in the anharmonic oscillator case [88, 91].

This suggests a connection between the crossover observed in the behaviour of the perturbative coefficients and the crossover behaviour of the RG flows. When we construct the RG flows starting near the Gaussian fixed point and let them evolve towards the high-temperature fixed point, it should be possible to describe the first iterations using the Gaussian scaling variables (see section 10). On the other hand, after a large number of iterations, the scaling variables of the HT fixed point are the relevant ones. If we use regular perturbation theory, we expect that it will be impossible to find a region where the two expansions are valid due to the zero radius of convergence of the weak-coupling expansions. On the other hand, if a field cutoff is introduced, the weak series have a nonzero radius of convergence and the direct calculations of critical amplitude as in [79] might be possible. A generic feature that we then expect is that if we calculate the perturbative coefficients with a field cutoff, by blockspinning, the first coefficients should stabilize quickly, while the large order in perturbation should stabilize after more iterations. This property was verified in [91].

8.3. Improved perturbative methods

The field cutoff significantly alter the accuracy of the perturbative series. This is illustrated in figure 4 where the accuracy of perturbation theory for the two-point function at various order is shown in regular perturbation theory and with a particular field cutoff. The figure makes clear that at sufficiently large coupling, the modified series becomes more accurate than the regular series. It is also clear that for a given field cutoff, the accuracy peaks near a specific region of the coupling. It is likely that at a given coupling, it is possible to find an optimal field cutoff that can be determined approximately using a strong-coupling expansion as in a simple integral discussed in [92].

8.4. Large field cutoff in ERGE

Understanding and controlling the large field configurations is an issue that goes beyond the scope of perturbation theory. In particular, it appears in the context of the RG flows of the

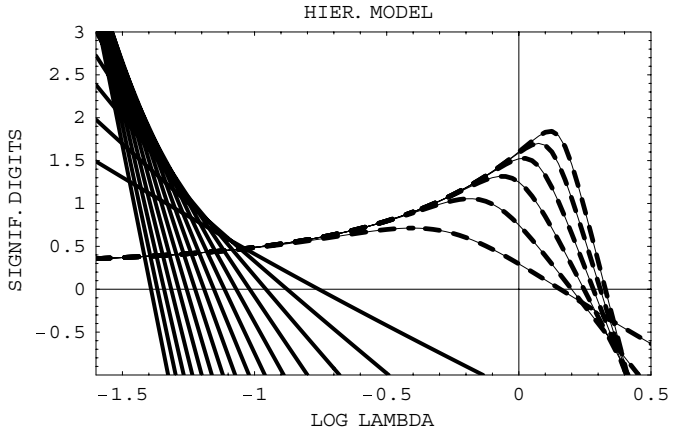

Figure 4. Number of significant digits obtained with regular perturbation theory at order $1, 3, 5, \ldots, 15$ (solid, turning clockwise with order) and with $\phi_{\max} = 2$ (dash line), at order $1, 3, \ldots, 11$ (moving up with order) as a function of λ, for the two-point function of the HM.

effective actions [93]. It was noted [94] that the introduction of a background field suppresses large field contributions to the flows. We should also mention functional generalizations of the Callan–Symanzik equation [95, 96] where a running bare mass controls large fluctuations.

9. Relation with the ERGE in the LPA approximation

9.1. Polchinski equation in the LPA

As explained in the first sections, the RG transformation of the HM can be reduced to a simple integral equation because of the very special form of the non-local interactions. In general, the real space RG seems difficult and one may prefer a formulation in terms of the Fourier transform of the fields. An UV cutoff can be introduced and lowered in variety of ways (sharp cutoff or smooth cutoff functions, ...). An effective action can be obtained by lowering the cutoff, or varying a parameter in the cutoff function in such a way that the large momentum components of the fields get more integrated. The derivative of the effective action with respect to the cutoff (or a related parameter) can then be expressed in terms of an integral over the momenta of a function of the action and its derivatives with respect to the Fourier transform of the fields. This equation is exact and is often called an exact renormalization group equation (ERGE). This idea was introduced and developed in [4, 11, 97, 98] and has generated a large interest that is still ongoing. Progress has been reviewed for instance in [8–10, 99, 100]. An ERGE can be rewritten as an infinite set of coupled partial differential equations. A simple starting point is to neglect the evolution of the terms in the effective action involving derivatives. This is called the local potential approximation (LPA).

A simple equation that can be written in this approximation for an ERGE with a smooth cutoff function is [98]:

$$\frac{\partial V}{\partial t} = DV + \left(1 - \frac{D}{2}\right)\phi\frac{\partial V}{\partial \phi} - \left(\frac{\partial V}{\partial \phi}\right)^2 + \frac{\partial^2 V}{\partial \phi^2}, \tag{9.1}$$

where $V(t, \phi)$ is the effective potential and t is a parameter that increases when the cutoff decreases.

9.2. Infinitesimal form of Gallavotti's recursion formula

We now consider the extension of Gallavotti's recursion formula equation (4.10). We introduce the notations

$$\mathcal{G}_n(\phi) = \mathrm{e}^{-V(\Lambda,\phi)} \tag{9.2}$$

$$\mathcal{G}_{n+1}(\phi) = \mathrm{e}^{-V(\Lambda/\ell,\phi)}. \tag{9.3}$$

In order to take the limit $\ell \to 1$, we need to fix the variance of the Gaussian weight and the overall normalization. We rewrite the extension of equation (4.10) for an arbitrary ℓ with a Gaussian weight parametrized in terms of a function $\rho(\ell)$ as

$$\exp(-V(\Lambda/\ell,\phi)) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\xi}{\sqrt{\pi\rho(\ell)}} \exp\left(-\frac{\xi^2}{\rho(\ell)} - \ell^D V(\Lambda, \ell^{1-D/2}\phi + \xi)\right). \tag{9.4}$$

We require that $\rho(1) = 0$ and $\rho'(1) \neq 0$, otherwise ρ is arbitrary. In [13], we have $\rho(\ell) = 2(\ell - 1)$.

The factor $(\pi\rho(\ell))^{-1/2}$ guarantees that when $\ell \to 1$, the two sides of equation (9.4) are equal. We now write $\ell = 1 + \delta$ and expand in δ. As mentioned, the terms of order 0 cancel. Terms of order $\delta^{1/2}$ appear but become zero after integration over ξ. Equating the terms of order δ, we obtain

$$-\Lambda\frac{\partial V}{\partial \Lambda} = DV + \left(1 - \frac{D}{2}\right)\phi\frac{\partial V}{\partial \phi} - \frac{\rho'(1)}{4}\left[\left(\frac{\partial V}{\partial \phi}\right)^2 - \frac{\partial^2 V}{\partial \phi^2}\right]. \tag{9.5}$$

If we write $\Lambda = \mathrm{e}^{-t}$ and change the ϕ scale in order to get rid of the factor $\frac{\rho'(1)}{4}$, we recover equation (9.1).

It should be pointed out that despite the fact that the differential equation comes as the coefficient of the order δ in the expansion of equation (9.4), equation (9.5) is not a linearization of equation (9.4). Indeed, when we reabsorb ρ in the integration variable ξ, we introduce a term of order $\sqrt{\rho}$ in the argument of the exponential. The terms that survive integration are quadratic in $\sqrt{\rho}$, namely the second derivative of u and the obviously nonlinear square of the first derivative of u. This point was not fully understood in [18].

Note also that the Gaussian fixed point corresponds to $V = 0$. The problem of finding the eigenvalues of the linearized RG transformation reduces to a time-independent Schrödinger problem. The $\phi\partial/\partial\phi$ term can be eliminated by minimal substitution, introducing a ϕ^2 term, and the problem can be mapped into the problem of finding the eigenvalues of an harmonic oscillator.

9.3. The critical exponents of Polchinski's equation

A related equation is the so-called Polchinski equation. It can be written for N components as [101]:

$$\frac{\partial u}{\partial t} = \frac{2y}{N}u'' + \left(1 + \frac{2}{N} + (2-d)y - 2yu\right)u' + (2-u)u, \tag{9.6}$$

with $y = \vec{\phi}\cdot\vec{\phi}$, $u = 2V'$ and the prime denotes derivatives with respect to y. This equation can be derived [101, 102] from an ERGE due to Polchinski [11] using the LPA. For $N = 1$, one can see that it follows from equation (9.1), by re-expressing it in terms of y and its derivatives and taking the derivative with respect to y of the resulting equation. Equation (9.1) is also obtained as the LPA of an ERGE due to Wilson [8]. The exponents were calculated in [101].

In particular for $N = 1$, they found $\gamma = 1.2992$ which is close to the HM value. A more precise value $\gamma = 1.299\,12$ was obtained in [17].

In addition, Litim [15, 57, 103] proposed an optimized ERGE and suggested [93, 104] that it was equivalent to the Polchinski equation in the local potential approximation. The equivalence was subsequently proved by Morris [105]. The value of γ for the optimized ERGE [57] in the case $N = 1$ is $1.299\,124$ and differ by 2 in the fifth decimal from the HM. More recently, [19] the calculations using the optimized ERGE and Polchinski equation were both repeated with more accuracy and compared. The numerical difference between the exponents of the two (analytically equivalent) formulations was reduced to 10^{-14}. Their final result is $\gamma = 1.299\,123\,547\,7613$ which confirms the non-equivalence with the HM. This question is also discussed for $N > 1$ in section 13.

9.4. Infinitesimal form of Wilson approximate recursion formula

The infinitesimal form of Wilson approximate recursion formula can be derived by following the same steps as for Gallavotti's recursion formula. First, we write $H_n[\phi] = \exp(-Q(\phi))$. We then use the arbitrariness of the scale of the fluctuations ξ as previously. The only difference is that the term of order $\sqrt{\delta}$ disappears from

$$Q((1+\delta)^{1-D/2}\phi + \sqrt{\delta}\xi) + Q((1+\delta)^{1-D/2}\phi - \sqrt{\delta}\xi). \tag{9.7}$$

Consequently, there seems to be no $(\partial Q/\partial\phi)^2$ term in the final equation. This point was also noted in [36]. This suggests that the limit $\ell \to 1$ is Gaussian ($\gamma = 1$). In [53], a numerical calculation of γ was done for values of $\ell = 2^{\zeta}$. $\zeta = 1$ corresponds to Wilson's case while $\zeta = 1/3$ corresponds to the HM. The limit we are interested in for the infinitesimal form is $\zeta \to 0$. Unfortunately, in this limit, the numerical procedure used in [53] becomes unstable because the errors in the integration routine become more important as we need to iterate more times the basic recursion formula. Figure 1 in [53] indicates that γ keeps increasing as ζ decreases. The last data point is for $\zeta = 0.3$. For smaller values of ζ, large errors bars develop as can be seen by repeating the calculation at closely chosen values of ζ. For instance near $\zeta = 0.15$, we found values of γ as high as 1.34 and as low as 1.28. This calculation should be repeated with more accurate integration methods.

9.5. Finite time singularities

It has been argued that some ERGE in the LPA have finite time singularities [106]. This is not surprising given that the solutions of the fixed point equation for equation (9.6) are generically singular. More precisely, if we assume that $\partial u/\partial t = 0$, we obtain a second-order differential equation for u. The solutions blow up at finite y for generic 'initial' conditions at $y = 0$. This means that the derivative of the potential becomes singular at a finite value of the field. It has been shown numerically [101] that regular solutions can be obtained for special initial values using the shooting method and that these solutions correspond to the nontrivial fixed point obtained with other formulations. Rigourous results concerning the existence of global stationary solutions of equation (9.4) can be found in [13].

On the other hand, the possibility of having finite time singularities in Monte Carlo RG calculations for nearest neighbour models is controversial [107, 108]. More generally, these reviews question the existence of renormalized or effective Gibbs measures defined by certain RG procedures.

For the HM with the polynomial approximation discussed in section 7, singularities after a finite number of iterations cannot appear (as they cannot appear for a finite-dimensional

quadratic map). More generally, it seems possible to prove the boundedness of $\mathcal{H}_n$ or $\mathcal{G}_n$ (defined in section 4), for finite n, for a large class of initial functions.

9.6. Improvement of the LPA

The improvement of the LPA for ERGE as a derivative expansion is a well-developed subject [8–10, 100]. However, progress is still needed in order to get estimates of the exponents which can compete in accuracy with the best methods available [35]. It is possible that the basic differential equations for the effective potential and the coefficients of terms involving derivatives of the fields could be worked backward, at finite ℓ, in order to produce a set of manageable coupled integral equations. The improvement of the hierarchical approximation is discussed with completely different methods in section 14. This could lead to $\ell \to 1$ equations that could be in turn compared with the existing ones. We hope that some communication between the two approaches will be developed in the future.

10. The nonlinear scaling fields

10.1. General ideas and definitions

In the study of ordinary differential equations, a standard method [109] to go beyond the linearized approximation near a fixed point consists in constructing new coordinates where the equations become linear. In the context of the RG method, these new coordinates are called the nonlinear scaling variables (or scaling fields) and were first introduced by Wegner [31, 110].

10.2. The small denominator problem

Rectification procedures are usually plagued with the 'small denominator problem' initially encountered by Poincaré in his study of perturbed integrable Hamiltonians [111]. In the RG case, this question needs to be discussed for each fixed point separately. To the best of our understanding, for the HM in $D = 3$, the problem can be completely avoided (but in a non-obvious way) for the HT fixed point, it is not present for the nontrivial fixed point and it is essential to generate logarithmic corrections to the scaling laws near the Gaussian fixed point [31, 112].

In the rest of this section, we will discuss in detail the case of the scaling variables associated with the HT fixed point and the nontrivial fixed point. Later we show that they can be combined in order to calculate non-universal critical amplitudes. The way the small denominator problem can be avoided for the scaling variables of the HT fixed point is interesting. At first sight, the construction seems impossible for $D = 3$ and more generally for rational values of D, because some of the denominators are exactly zero. A numerical study in $D = 3$ showed [113] that for all zero denominators considered, a zero numerator miraculously appears. Explicit calculations in arbitrary dimensions and general arguments explaining why it should work to all orders were given in [114] and are summarized in subsection 10.5.

10.3. The linear scaling variables of the HT fixed point

As explained in section 7, the RG transformation in the symmetric phase can be approximated very accurately in terms of a quadratic map in a $(l_{\max})$-dimensional space

$$a_{n+1,l} = \frac{u_{n,l}}{u_{n,0}}, \tag{10.1}$$

with

$$u_{n,\sigma} = \Gamma_{\sigma}^{\mu\nu} a_{n,\mu} a_{n,\nu}, \tag{10.2}$$

and

$$\Gamma_{\sigma}^{\mu\nu} = (c/4)^{\mu+\nu} \frac{(-1/2)^{\mu+\nu-\sigma}(2(\mu+\nu))!}{(\mu+\nu-\sigma)!(2\sigma)!}, \tag{10.3}$$

for $\mu + \nu \geqslant \sigma$ and zero otherwise. As in 'relativistic' notations, the Greek indices μ and ν go from 0 to $l_{\max}$, while Latin indices i, j go from 1 to $l_{\max}$. Repeated indices mean summation unless specified differently. With the normalization of equation (10.1), $a_n, 0 = 1$ for any n and is not a dynamical variable. For small departure from the HT fixed point $\delta a_{n,i}$, the linear RG transformation reads

$$\delta a_{n+1,i} \simeq \mathcal{M}_i^j \delta a_{n,j}, \tag{10.4}$$

with

$$\mathcal{M}_i^j = 2\Gamma_i^{j0} = 2\left(\frac{c}{4}\right)^j \left(-\frac{1}{2}\right)^{j-i} \frac{(2j)!}{(2i)!(j-i)!}, \tag{10.5}$$

for $i \leqslant j$ and zero otherwise.

$\mathcal{M}$ is of upper triangular form and the spectrum is given by the diagonal elements

$$\tilde{\lambda}_{(r)} = 2(c/4)^r = \ell^{D-r(D+2)} \tag{10.6}$$

in agreement with [30]. Note that the quantity in the exponent of ℓ also appears in the exponent γ_q when $q = 2r$ in equation (7.30). The tilde refers to the HT variables, eigenvalues etc... in order to avoid confusion with the same quantities for the nontrivial fixed point. As we assume $c < 2$ in order to have a well-defined infinite volume limit (see section 6), all the eigenvalues are less than 1 and the fixed point is completely attractive. As r increases, the eigenvalues decrease and become more irrelevant. Equation (10.6) has a simple interpretation: 2 stands for the volume increase and the $(c/4)^r$ for the rescaling of the $2r$ th power of the sum of the fields, which has a dimension $a^{D/2+1}$ in lattice spacing units, when the volume element a^D is properly included, and of course assuming $\eta = 0$. The eigenvalues in equation (10.6) can also be seen in equation (7.23)

We call $\mathcal{R}$ the matrix of right eigenvectors:

$$\mathcal{M}_l^i \mathcal{R}_i^r = \tilde{\lambda}_{(r)} \mathcal{R}_l^r, \tag{10.7}$$

(with no summation over r). For convenience, the columns of $\mathcal{R}$ are ordered as the eigenvalues. We introduce the linear coordinates $\tilde{h}_l$:

$$a_{n,l} = \mathcal{R}_l^r \tilde{h}_{n,r}, \tag{10.8}$$

and which transform as

$$\tilde{h}_{n+1,r} \simeq \tilde{\lambda}_{(r)} \tilde{h}_{n,r} \tag{10.9}$$

in the linear approximation. The matrix $\mathcal{R}_i^r$ and its inverse are also upper triangular and $\tilde{h}_l$ is of order β^l, just as $a_{n,l}$ is. We fix the normalization of the right eigenvectors in $\mathcal{R}$ in such a way that all the diagonal elements are 1. This guarantees that $\tilde{h}_l = a_{n,l} + \mathcal{O}(\beta^{l+1})$. In [114], it was proved that for the upper diagonal elements ($j > i$),

$$\mathcal{R}_i^j = \left(\frac{-c}{8-2c}\right)^{j-i} \frac{(2j)!}{(2i)!(j-i)!}, \tag{10.10}$$

and that

$$(\mathcal{R}^{-1})_i^j = (-1)^{j-i} \mathcal{R}_i^j. \tag{10.11}$$

Note that because of the HT selection rules (i.e., the upper triangular form of the matrices), the matrix elements do not depend on the choice of $l_{\max}$.

It is easy to rewrite the exact RG transformation in the $\tilde{h}_l$ coordinates. Starting with the basic equation (10.1), we replace a_0 by 1 and a_l by $\mathcal{R}_l^p \tilde{h}_p$, we obtain a recursion formula of the form

$$\tilde{h}_{n+1,l} = \frac{\tilde{\lambda}_{(l)}\tilde{h}_l + \Delta_l^{pq}\tilde{h}_p\tilde{h}_q}{1 + 2\Delta_0^{p0}\tilde{h}_p + \Delta_0^{pq}\tilde{h}_p\tilde{h}_q}, \tag{10.12}$$

with coefficients calculable from equation (10.3). For instance,

$$\Delta_l^{pq} = (\mathcal{R}^{-1})_l^{l'}\Gamma_{l'}^{p'q'}\mathcal{R}_{p'}^{p}\mathcal{R}_{q'}^{q}.$$

Pursuing the relativistic analogy, upper roman indices transform with $\mathcal{R}$ and the lower ones with $(\mathcal{R})^{-1}$.

10.4. The nonlinear scaling variables of the HT fixed point

We now explain how to re-express the linear variables $\tilde{h}_l$ in terms of the nonlinear scaling variables $\tilde{y}_l$ for which the approximate multiplicative transformation of equation (10.9) is assumed to be exact

$$\tilde{y}_{n+1,r} = \tilde{\lambda}_{(r)}\tilde{y}_{n,r}. \tag{10.13}$$

If we use $ln(y_l)$ as our new coordinates, the RG flows become parallel straight lines. All the dynamics is then contained in the mapping. The monotonicity of these functions suggests a possible connection with field theory entropy [115]; however the regularity near other fixed points may be an issue.

As in [31, 110], we introduce the expansion

$$\tilde{h}_l = \tilde{y}_l + \sum_{i_1,i_2,\ldots} s_{l,i_1 i_2\ldots}\tilde{y}_1^{i_1}\tilde{y}_2^{i_2},\ldots, \tag{10.14}$$

where the sums over the i's run from 0 to infinity in each variable with at least two non-zero indices. In the following, we use the notation $\mathbf{i}$ for $(i_1, i_2, \ldots)$. More generally, such vectors will be represented by boldface characters. The unknown coefficients $s_{l,\mathbf{i}}$ in equation (10.14) are obtained by matching two expressions of $\tilde{h}_{+1,l}$, one obtained from the RG transformation of the h_l given in equation (10.12), the other obtained by evolving the scaling variables according to the exact multiplicative transformation equation (10.13). The matching conditions can be written as

$$\tilde{h}_{n+1,l}(\tilde{\mathbf{h}}_n(\tilde{\mathbf{y}})) = \tilde{h}_l(\tilde{\lambda}_1\tilde{y}_{n,1}, \tilde{\lambda}_2\tilde{y}_{n,2}, \ldots), \tag{10.15}$$

and yield the conditions

$$s_{l,\mathbf{i}} = \frac{N_{l,\mathbf{i}}}{D_{l,\mathbf{i}}}, \tag{10.16}$$

with

$$N_{l,\mathbf{i}} = \sum_{\mathbf{j}+\mathbf{k}=\mathbf{i}}\left(-\Delta_l^{pq}s_{p,\mathbf{j}}s_{q,\mathbf{k}} + s_{l,\mathbf{j}}\prod_m \tilde{\lambda}_{(m)}^{j_m}2\Delta_0^{p0}s_{p,\mathbf{k}}\right) + \sum_{\mathbf{j}+\mathbf{k}+\mathbf{r}=\mathbf{i}} s_{l,\mathbf{j}}\prod_m \tilde{\lambda}_{(m)}^{j_m}\Delta_0^{pq}, s_{p,\mathbf{k}}s_{q,\mathbf{r}} \tag{10.17}$$

and

$$D_{l,\mathbf{i}} = \tilde{\lambda}_{(l)} - \prod_m \tilde{\lambda}_{(m)}^{i_m}. \tag{10.18}$$

For a given set of indices $\mathbf{i}$, we introduce the notation

$$\mathcal{I}_q(\mathbf{i}) \equiv \sum_m i_m m^q. \tag{10.19}$$

One sees that $\mathcal{I}_0$ is the degree of the associated product of scaling variables and $\mathcal{I}_1$ its order in the HT expansion (since y_l is also of order β^l). Given that all the indices are positive and that at least one index is not zero, one can see that if $\mathbf{j}+\mathbf{k}=\mathbf{i}$ then $\mathcal{I}_q(\mathbf{j}) < \mathcal{I}_q(\mathbf{i})$ and $\mathcal{I}_q(\mathbf{k}) < \mathcal{I}_q(\mathbf{i})$. Consequently, equation (10.17) yields a solution order by order in $\mathcal{I}_0$ or in $\mathcal{I}_1$ (since the rhs always contains $s_{l,\mathbf{i}}$ of lower order in $\mathcal{I}_0$ and $\mathcal{I}_1$) provided that none of the denominators $D_{l,\mathbf{i}}$ are exactly zero.

We can now rewrite the denominators as

$$D_{l,\mathbf{i}} = 2\left(\frac{c}{4}\right)^l - 2^{\mathcal{I}_0(\mathbf{i})}\left(\frac{c}{4}\right)^{\mathcal{I}_1(\mathbf{i})}. \tag{10.20}$$

The parametrization of c in equation (3.23) implies that a zero denominator appears when

$$D - l(D+2) = D\mathcal{I}_0(\mathbf{i}) - (D+2)\mathcal{I}_1(\mathbf{i}). \tag{10.21}$$

Given that the $\mathcal{I}_q$ are integers, this can only occur at some rational values of D. Ignoring temporarily this set of values, we can say that for generic values of c, the denominators are not zero. In the spirit of dimensional regularization, we can perform, order by order in the HT order $\mathcal{I}_1$, the construction of the $s_{l,\mathbf{i}}$ for a generic value of c and discuss the limit where c takes some special value at the end of the calculation. Since the linear problem is completely solved and we may assume $\mathcal{I}_0(\mathbf{i}) > 1$. In addition, since both h_l and y_l are of order β^l, we need $\mathcal{I}_1(\mathbf{i}) \geqslant l$. At lowest nontrivial order in β, we have $\mathcal{I}_l(\mathbf{i}) = l$, and it has been shown [114] that, at that order, $s_{l,\mathbf{i}}$ has only an apparent pole of order l at $c=0$ exactly cancelled by a zero of the same order in the numerator. If $\mathcal{I}_1(\mathbf{i}) > l$, we can write

$$D_{l,\mathbf{i}} = 2\left(\frac{c}{4}\right)^l (c_{\text{crit}})^{l-\mathcal{I}_1(\mathbf{i})} T_{l,\mathbf{i}}, \tag{10.22}$$

with

$$T_{l,\mathbf{i}} = \left(c_{\text{crit}}^{\mathcal{I}_1(\mathbf{i})-l} - c^{\mathcal{I}_1(\mathbf{i})-l}\right), \tag{10.23}$$

and

$$c_{\text{crit}} = 4 \times 2^{(1-\mathcal{I}_0(\mathbf{i}))/(\mathcal{I}_1(\mathbf{i})-l)}. \tag{10.24}$$

The only poles that we need to worry about are those where $0 < c_{\text{crit}} < 2$. An inspection [114] of the 175 terms up to order β^7 shows that all the poles at $0 < c < 2$ were exactly cancelled by zeros of the same order. Note that the maximally simplified rational expression for the $s_{l,\mathbf{i}}$ do have poles but at values of c outside of the range $0 < c < 2$. A possible strategy for a proof would be to show inductively that for each term with $0 < c_{\text{crit}} < 2$, the cancellation occurs and so the undesired poles do not propagate to higher order. Such an algebraic proof seems difficult because the $s_{l,\mathbf{i}}$ are rational expressions and the zeros at the numerator only appear after factorization of sums of such terms. It seems nevertheless reasonable to conjecture that $s_{l,\mathbf{i}}$ have no poles $0 < c_{\text{crit}} < 2$. If this conjecture is correct, dimensional regularization provides a unique continuous expression for the coefficients for any c with $0 < c < 2$, and the model is formally 'solvable' using the recursion for the coefficients given by equation (10.17). The conjecture implies that for any value of c in this interval, we can construct analytical expression of $a_{n,l}$ (which contains all the thermodynamical quantities) in terms of $a_{0,l}$ (which depends on the initial energy density):

$$a_{n,l} = (\mathcal{R}^{-1})^r_l \tilde{h}_r\left(\tilde{\lambda}_1^n \tilde{y}_1(\mathbf{a}_0), \tilde{\lambda}_2^n \tilde{y}_2(\mathbf{a}_0), \dots\right). \tag{10.25}$$

We will see that the initial values of $\mathbf{y}(\mathbf{a}_0)$ have a simple interpretation given in equation (10.41).

It is also possible to express the nonlinear scaling variables in terms of the linear variables. Writing

$$\tilde{y}_l = \tilde{h}_l + \sum_{\mathbf{i}} r_{l,\mathbf{i}} \prod_m \tilde{h}_m^{i_m}, \tag{10.26}$$

we can determine order by order the unknown coefficients $r_{l,\mathbf{i}}$ of the expansion for generic values of c.

10.5. Argument for the cancellation to all orders

The generating function of the connected parts of the average values of the total field reads

$$\ln(R_n(k)) = a^c_{n,1}k^2 + a^c_{n,2}k^4 + \cdots, \tag{10.27}$$

with

$$a^c_{n,l} = \sum_{\mathbf{i}:\mathcal{I}_1(\mathbf{i})=l} (-1)^{\mathcal{I}_0(\mathbf{i})-1}(\mathcal{I}_0(\mathbf{i})-1)! \prod_m \frac{a_m^{i_m}}{i_m!}. \tag{10.28}$$

We are working in the HT phase and that we do not need to subtract powers of the magnetization. After a suitable rescaling of k described in subsection 7.1, we have

$$a^c_{n,l} = (-\beta)^l \frac{1}{2l!}\left(\frac{c}{4}\right)^{ln} \langle(\phi_n)^{2l}\rangle^c. \tag{10.29}$$

We assume that the initial values $a_{0,l}$ are such that

$$\lim_{n\to\infty} \chi_n^{(q)} = \chi^{(q)} \tag{10.30}$$

is finite. In other words, we assume that $\beta < \beta_c$ and that all the $\chi^{(q)}$ are finite. As explained in section 6, this statement can be proved rigorously for a Ising measure. From equation (10.29), it is then clear that for n large enough, we have the leading scaling

$$a^c_{n,l} \propto \left(2\left(\frac{c}{4}\right)^l\right)^n = \tilde{\lambda}^n_{(l)}. \tag{10.31}$$

This suggests a simple relationship between $a^c_{n,l}$ and $\tilde{y}_{n,l}$. Using the Möbius inversion formula [116, 117], it has been shown that

$$a^c_l = \tilde{y}_l + \mathcal{O}(\beta^{l+1}). \tag{10.32}$$

Equation (10.32) means that there are no nonlinear contributions of order β^l to a^c_l. For instance, there are no y_1^3 or $y_1 y_2$ terms in a^c_3. This is expected because the nonlinear terms of order β^l scale faster than y_l, (assuming $0 < c < 2$). By saying a term 'scale faster', we mean that it goes to zero at a slower rate when n becomes large. In general, at each RG step, a term $\prod_m y_m^{i_m}$ of order β^l is multiplied by

$$2^{\mathcal{I}_0(\mathbf{i})}\left(\frac{c}{4}\right)^l > \tilde{\lambda}_{(l)} = 2\left(\frac{c}{4}\right)^l.$$

The strict inequality comes from the fact that for the nonlinear terms $\mathcal{I}_0(\mathbf{i}) > 1$. It is thus clear that nonlinear terms of order β^l would spoil the HT scaling of equation (10.31) and contradict the existence of a infinite volume limit.

For higher order terms, the sign of the denominator $D_{l,\mathbf{i}}$ introduced in equation (10.18) tells us whether or not the term scales faster or slower than the linear term. With our sign convention, $c > c_{\text{crit}}(l,\mathbf{i})$, means $D_{l,\mathbf{i}} < 0$ and the term spoils the HT scaling

equation (10.31). Since the coefficients are rational functions of c, they cannot vanish suddenly when c becomes larger than $c_{\rm crit}(l, \mathbf{i})$. Consequently, whenever $0 < c_{\rm crit}(l, \mathbf{i}) < 2$, the coefficient of the corresponding term is expected to vanish identically.

We have checked that this argument is consistent with our previous explicit calculations. We have used equations (10.28), (10.8) and the already calculated coefficients in equation (10.14) to calculate

$$a_l^c = \tilde{y}_l + \sum_{\mathbf{i}:\mathcal{I}_1(\mathbf{i})>l} t_{l,\mathbf{i}} \tilde{y}_1^{i_1} \tilde{y}_2^{i_2} \dots, \tag{10.33}$$

up to order 7. For all the 50 terms with $0 < c_{\rm crit} < 2$, the corresponding $t_{l,\mathbf{i}}$ are identically zero.

The existence of an infinite volume limit implies that the small denominator problem can be evaded for any c such that $0 < c < 2$. We have constructed the a_l^c in terms of the a_l. However, we could have proceeded directly, writing the $a_{n+1,l}^c$ in terms of the $a_{n,l}^c$:

$$a_{n+1,l}^c = \mathcal{M}_l^k a_{n,k}^c + \sum_{k+q \geqslant l} v_l^{kq} a_{n,k}^c a_{n,q}^c + \cdots. \tag{10.34}$$

The coefficients v_l^{kq} and the higher order ones can be obtained by using the expansion of equation (10.27) in the logarithm of equation (3.12) and expanding order by order in $\mathbf{a}_n^c$. The series does not terminate. The linear transformation is the same as before because a_l^c and a_l only differ by nonlinear terms. Using

$$a_{n,l}^c = \mathcal{R}_l^r \tilde{h}_r^c, \tag{10.35}$$

we obtain

$$\tilde{h}_{n+1,l}^c = \tilde{\lambda}_{(l)} \tilde{h}_{n,l}^c + \sum_{k+q \geqslant l} w_l^{kq} \tilde{h}_{n,k}^c \tilde{h}_{n,q}^c + \cdots. \tag{10.36}$$

We then introduce the expansion

$$\tilde{h}_l^c = \tilde{y}_l + \sum_{\mathbf{i}:\mathcal{I}_1(\mathbf{i})>l} s_{l,\mathbf{i}}^c \prod_m \tilde{y}_m^{i_m}, \tag{10.37}$$

and obtain

$$s_{l,\mathbf{i}}^c = \frac{N_{l,\mathbf{i}}^c}{D_{l,\mathbf{i}}}, \tag{10.38}$$

with $N_{l,\mathbf{i}}^c$ given by a formula similar to equation (10.17), except that it does not terminate. A detailed analysis shows that the two formulae have in common that the numerator depends only on coefficients of strictly lower orders in β, and equation (10.38) can be used order by order in β to construct the $s_{l,\mathbf{i}}^c$ for generic values of c.

Since $\mathcal{R}^{-1}$ is upper triangular, we see from equation (10.35) that $\tilde{h}_l^c$ is equal to a_l^c plus terms which go to zero faster. Consequently, for large n, the leading scaling is

$$\tilde{h}_{n,l}^c \propto \tilde{\lambda}_{(l)}^n. \tag{10.39}$$

Following reasonings used before, this implies that terms in the expansion equation (10.37) that scale faster than y_l for any $0 < c < 2$ should have a vanishing coefficient. In other words,

$$0 < c_{\rm crit}(l, \mathbf{i}) < 2 \quad \Rightarrow \quad s_{l,\mathbf{i}}^c = 0.$$

Given the specific form of the $s_{l,\mathbf{i}}^c$ given in equation (10.38), the $\tilde{h}_l^c$ have no poles for $0 < c < 2$. The a_l^c being linear combinations of $\tilde{h}_l^c$ and the a_l being linear combinations of products of a_l^c, we conclude that the expansion of the a_l in terms of the scaling variables has also no poles for $0 < c < 2$.

Again we see that there exists a unique continuous definition of the scaling variables that can be used at particular values of c where the denominator is exactly zero. From a practical point of view, the calculation at fixed c of the $s^c_{l,\mathbf{i}}$ is easier than the calculation of the $s_{l,\mathbf{i}}$, because no limit needs to be taken explicitly. The $s^c_{l,\mathbf{i}}$ being rational function of c cannot be zero everywhere except at isolated values. Consequently, we can set to zero the $s^c_{l,\mathbf{i}}$ having $c_{\text{crit}}(l,\mathbf{i}) < 2$ even at values of c, where $D_{l,\mathbf{i}} = 0$.

The initial values $\tilde{\mathbf{y}}_0$ have a very simple interpretation. We know that $\tilde{y}_{n,l}$ is the *only* leading term of $a^c_{n,l}$ when n becomes large. If at a given $0 < c < 2$, a nonlinear term scales exactly like $\tilde{y}_{n,l}$, then by increasing c slightly (but keeping $c < 2$), we can make this term dominant in contradiction with the existence of the infinite volume limit. Consequently,

$$\lim_{n\to\infty} \tilde{\lambda}_l^{-n} a^c_{n,l} = \lim_{n\to\infty} \tilde{\lambda}_l^{-n} \tilde{y}_{n,l} = \tilde{y}_{0,l}. \tag{10.40}$$

From equation (10.29), we see that

$$\tilde{y}_{0,l} = (-\beta)^l \frac{1}{2l!} \chi^{(2l)}. \tag{10.41}$$

Furthermore, [118] we can consider the $\chi^{(2l)}$ as functions of the initial values $a_{0,l}$. If we now replace these initial values by the m-advanced values $a_{m,l}$, we find that $\chi^{(2l)}(a_{m,l}) = \tilde{\lambda}_l^m \chi^{(2l)}(a_{0,l})$. In this sense, the infinite volume limit quantities $\chi^{(2l)}$ can be seen as scaling variables.

10.6. The nonlinear scaling variables of the nontrivial fixed point

The construction of the nonlinear scaling variables can be repeated verbatim [119] for the nontrivial fixed point. However, unlike the HT case, all the calculations have to be performed numerically. Starting from the basic quadratic map of equation (10.1), introducing new coordinates that are zero at the nontrivial fixed point, re-expressing these coordinates as linear combinations of the right eigenvectors and fixing the scale (for instance, by requiring that the HT fixed point is located at $(1, 1, \ldots)$), we obtain a RG transformation for the linear scaling variables (denoted h) that can be written in the form

$$h_{n+1,r} = \frac{\lambda_r h_{n,r} + \Delta_r^{pq} h_{n,p} h_{n,q}}{1 + \Lambda^p h_{n,p} + \Delta_0^{pq} h_{n,p} h_{n,q}}. \tag{10.42}$$

As in the HT case, the first subscript refers to the number of iterations and the second is the index of the variable.

As discussed in section 7, for $D = 3$ there is one and only one eigenvalue larger than 1. We can express the linear scaling variables in terms of the nonlinear scaling fields (denoted by y):

$$h_{n,r} = \sum_{\mathbf{i}} t_{r,\mathbf{i}} \prod_m y_{n,m}^{i_m}, \tag{10.43}$$

and proceed as before. One can also find expansions of the scaling fields in terms of the linear variables, by setting

$$y_{n,r} = \sum_{\mathbf{i}} u_{r,\mathbf{i}} \prod_m h_{n,m}^{i_m}. \tag{10.44}$$

The only potential problem comes from small denominators. In [119, 113], a partial numerical survey of possible small denominators was done. The worse case found was $\lambda_2^9 \simeq \lambda_4$ with two parts in a thousand.

A major difference with the HT case is the existence of a relevant direction. Consequently, y_1 plays a very special role as a coordinate along the unstable direction. As y_1 can be expressed

in terms of the linear scaling variables which are themselves linear combinations of the original coordinates **a**, we can have

$$y_1(\mathbf{a}) = 0 \tag{10.45}$$

as the equation defining the stable manifold. Similarly, the unstable manifold in the **a** coordinates corresponds to the one-dimensional trajectory $\mathbf{a}(y_1, 0, 0, \ldots)$. For practical purpose, one can expand the linear scaling variables to large order in y_1 and to low order in a few irrelevant variables.

10.7. Convergence issues

Up to now, it has been shown that it seems possible to construct formal expansion of the linear scaling variables in terms of the nonlinear scaling variables or vice versa, for the HT fixed point or the nontrivial fixed point. This does not mean that these expansions define analytical functions. In contrast, the existence of multiple fixed points suggests that these expansions have at best a finite radius of convergence. Numerical experiments [119] testing the scaling of the nonlinear scaling variables suggest that the two expansions have overlapping region of convergence. This will be illustrated in subsection 11.3.

10.8. The scaling variables of the Gaussian fixed point

The eigenvalues of the parity preserving linearized RG transformation at the Gaussian fixed point [30] are

$$\lambda_{Gj} = 2c^{-j} = \ell^{D-j(D-2)} \tag{10.46}$$

for $j = 1, 2, \ldots$. The interpretation is simply the scaling of the parity invariant couplings in $g_{2j}\phi^{2j}$ interactions. For $D \geqslant 4$, there is only one relevant direction corresponding to the mass term. For $2 < D < 4$, there are at least two and at most a finite number of relevant directions.

This problem can be reformulated in term of the evolution operator of the harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}, \tag{10.47}$$

during a finite Euclidean time $t = -\mathrm{i}\tau$. The correspondence is $c = \exp(2\omega\tau)$ and $m = \beta(c-1)/(2-c)$ in $\hbar = 1$ units. The connection with the $\ell \simeq 1 + \delta$ limit discussed in section 9 is $\omega\tau = \delta(D-2)/2$.

There are many zero denominators in integer dimensions, e.g., $\lambda_1 = \lambda_2^2$ for $D = 3$. If the numerators are not zero, one can modify [31] the situation by considering n-dependent coefficients. This generates logarithmic corrections which are necessary. These can be observed in the large order of the HT temperature expansion in [120]. Another possibility is to use the idea of dimensional regularization [121] as already explained in the HT case. This might help reinterpreting the connection between the $1/\epsilon$ poles in the $D = 4\text{-}\epsilon$ regularization and the logarithmic divergences in a cutoff regularization [122].

Practical constructions of the nonlinear scaling variables remain to be developed. As the radius of convergence of perturbative series is zero it is not clear that the procedure would work as in the two other cases. Modified perturbative methods where a cutoff in field space is introduced [85, 88, 91] might work better.

We are not aware of any explicit construction of the scaling variables for the low-temperature fixed point.

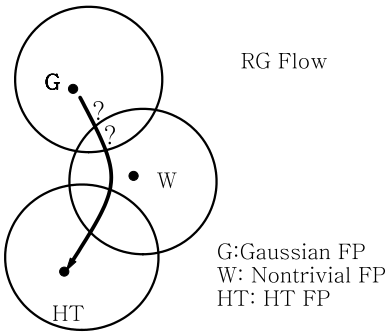

Figure 5. A qualitative description of a RG flow starting near the Gaussian fixed point, passing by the nontrivial fixed point and ending on the HT fixed point.

11. Interpolation between fixed points and critical amplitudes

11.1. Global RG flows

The critical amplitudes are in general non-universal, and their calculation requires that we go beyond the linear approximation and interpolate among fixed points. In the rest of this section, we consider mostly the flows from the nontrivial fixed point to the HT fixed point.

Given the success of field theoretical methods based on perturbation theory [23, 33, 34], it would certainly be desirable to extend the construction of nonlinear scaling variables with initial condition near the Gaussian fixed point. As explained in the previous section, this construction of the nonlinear scaling variables remains to be done. The basic picture is that the Gaussian fixed point is located on the stable manifold of the nontrivial fixed point and that it is possible, for $D < 4$, to use the unstable directions of the Gaussian fixed point to reach the nontrivial fixed point. For this reason, the Gaussian fixed point is often called the UV fixed point and the nontrivial fixed point the IR fixed point. The situation is depicted in figure 5.

11.2. Critical amplitudes and RG invariants

In section 10, we have seen that the construction of the coordinates $\mathbf{a}$ in terms of the nonlinear HT scaling variables provides a formal solution to the problem of the RG flows. However, this is not the end of the story since it is not clear how accurate finite-order expansions can be. Also the initial values were identified up to a factor $(-\beta)^r/2r!$ with the infinite volume susceptibilities $\chi^{(2r)}$, the very quantities that we would like to compute!

Indeed, our goal is to compute $\chi^{(2r)}$ or equivalently $\tilde{y}_{0,r}$ for initial conditions near the nontrivial fixed point where it is easy to use the other scaling variables $\mathbf{y}$. For simplicity, we will assume that the initial conditions are exactly on the unstable direction and close to the nontrivial fixed point:

$$y_{0,1} = u, \qquad y_{0,2} = 0, \qquad y_{0,3} = 0 \ldots.$$

More complicated cases are discussed in [119]. Under a RG transformation, $u \to \lambda_1 u$ while the HT nonlinear scaling variables transform multiplicatively according to equation (10.13). From equations (7.23) and (10.6), we can rewrite

$$\gamma_{2r} = -\frac{\ln \tilde{\lambda}_{(r)}}{\ln \lambda_1}. \tag{11.1}$$

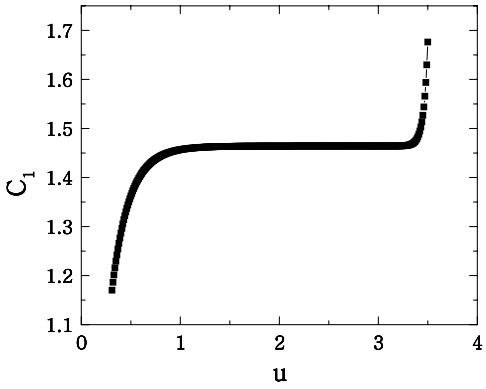

Figure 6. $C_1(u)$ defined in equation (11.4) versus u.

This implies that

$$C_r \equiv \tilde{y}_{n,r}(y_{n,1})^{\gamma_{2r}}, \tag{11.2}$$

are n independent or in other words, RG invariant. These relations suggest that the two fixed points are in some approximate sense dual [123] to each others. We can now rewrite the basic quantities that we want to calculate as

$$\tilde{y}_{0,r} = \tilde{y}_{0,r}(y_{0,1})^{\gamma_{2r}}(y_{0,1})^{-\gamma_{2r}} = C_r(u)^{-\gamma_{2r}}. \tag{11.3}$$

Constant C_r are not universal. They depend on the normalization choice for y, however, equation (11.3) is independent of this choice. It is possible [28] to relate u to $\beta - \beta_c$ for a particular model and consequently, calculating C_r provides an estimate of the critical amplitudes. The remaining question is: can we calculate C_r?

11.3. Overlapping regions of convergence

In principle, the RG invariants C_r can be calculated for any n. In practice, low n calculations fail because of the low accuracy of the HT expansion and large n calculations fail because of the low accuracy of the expansion in the nontrivial scaling variables. The accuracy of the two expansions in intermediate regions can be tested empirically by monitoring the stability of the estimates of C_r. In figure 6, we have displayed

$$C_1(u) = y_1(\mathbf{h}(\mathbf{h}_W(u))u^{\gamma}, \tag{11.4}$$

with the HT scaling variable $y_1(\mathbf{h})$ calculated up to order β^{11} and $\mathbf{h}_W(u)$ calculated up to order 80 in u. The RG invariance only implies that $C_1(\lambda u) = C_1(u)$ (we remind that $\lambda \simeq 1.427$). A wide plateau is observed in figure 6 for $1 < u < 3$, indicating good convergence properties in overlapping regions. RG invariance only forces periodicity on a log scale, but apparently the log-periodic oscillations are very small. They can be seen better in simplified models [28]. Log-periodic oscillation is discussed into more detail below in subsection 11.5.

11.4. Approximately universal ratios of amplitudes

Using the first $l_{\max}$ HT nonlinear scaling variables, it is possible to construct $l_{\max} - 1$ constants of motion

$$G_r \equiv -(2r)!\frac{\tilde{y}_{n,r}}{(-2\tilde{y}_{n,1})^{(r-1)(D/2)+r}}. \tag{11.5}$$

These quantities are RG invariants. We can evaluate them at $n = 0$. Using equation (10.41), we obtain

$$G_r = (-1)^{r+1} \frac{\beta^{\frac{D}{2}(1-r)} \chi^{(2r)}}{(\chi^{(2)})^{(r-1)(D/2)+r}}. \tag{11.6}$$

If the conjecture [124] that $(-1)^{r+1}\chi^{(2r)} > 0$ is correct, then $G_r > 0$.

We now evaluate the universal ratio on the unstable direction of the unstable fixed point. Calling $G_r(u)$ the corresponding value, we have

$$G_r(\lambda u) = G_r(u). \tag{11.7}$$

Consequently, we have the Fourier expansion

$$G_r(u) = \sum_q A_{r,q} u^{\mathrm{i}q\omega} = \sum_q A_{r,q}\, \mathrm{e}^{\mathrm{i}q\omega \ln u}, \tag{11.8}$$

with

$$\omega = \frac{2\pi}{\ln \lambda_1}. \tag{11.9}$$

The function is clearly periodic in $\ln(u)$, and we call the oscillations due to the non-zero Fourier modes 'log-periodic'. The coefficients can be calculated as

$$A_{r,q} = \frac{1}{\lambda_1} \int_1^{\lambda_1} \frac{\mathrm{d}u}{u} u^{-\mathrm{i}q\omega} G_r(u). \tag{11.10}$$

The oscillatory terms are very small, as noted in [27, 28, 123], and we have the approximate universal ratios

$$G_r(u) \simeq A_{r,0}. \tag{11.11}$$

These constants can be estimated using the methods discussed in subsection 11.3. The smallness of the nonzero Fourier modes also applies the non-universal function $C_l(u)$ discussed in subsection 11.3.

11.5. More about log-periodic corrections

The possibility of log-periodic terms was first discussed in [32, 45]. They were identified in the high-temperature expansion [27, 28] of the HM. The amplitudes $A^{(2l)}_{per.}$ are however quite small, typically, they affect the 16th significant digit of the susceptibility, and it takes a special effort to resolve them numerically. They are amplified [27, 28] by estimators of critical exponents such as the extrapolated slope $\hat{S}_m$ designed [125] to remove subleading corrections in estimation of the critical exponents at successive order in the HT expansion. This prevents an accurate determination of γ from HT expansion [27, 28].

Limit cycles often appear in two-dimensional ordinary differential equations. However, their stability in higher dimensions is an issue debated in the dynamical system community. For instance, the Landau scenario for the onset of turbulence, based on the appearance of limit cycles, is not considered viable. For the case of interest here, the log-periodic oscillations reflect the discrete invariance of the original Hamiltonian of equation (3.1). This symmetry protects the periodicity even in the continuum limit. However, this remnant of the discrete structure is very small numerically.

It is possible to design toy models [28] where the effect is larger for instance, the quadratic map

$$\tilde{h}_{n+1} = \xi \tilde{h}_n + (1-\xi)\tilde{h}_n^2, \tag{11.12}$$

for small values of ξ. The first nonzero Fourier mode is quite visible in the analogue of C_1 defined in equation (11.4) as shown in figure 7 for $\xi = 0.1$.

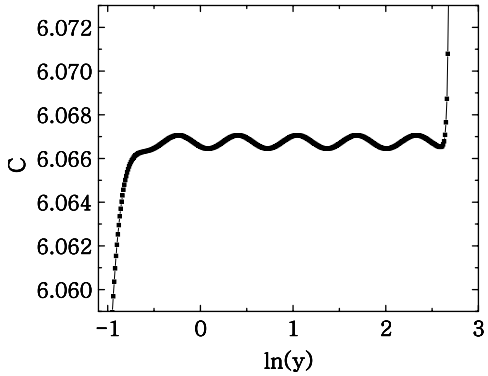

Figure 7. The analogue of C_1 in figure 6, for the quadratic map of equation (11.12) with $\xi = 0.1$, versus the natural logarithm of the scaling variable y.

12. Nontrivial continuum limits

12.1. The infinite cutoff limit

In this section, we apply the general procedure outlined by Wilson in [32] for the approximate recursion formula of equation (5.1) to the HM. We consider a sequence $K = 1, 2, \ldots$ of models with $\beta = \beta_c - \lambda_1^{-K} u$, where u is positive but not too large and λ_1 is the only relevant eigenvalue as in section 11. β_c depends on the particular choice of initial local measure W_0. We introduce the increasing sequence of UV cutoffs

$$\Lambda_K = 2^{\frac{K}{D}} \Lambda_R, \tag{12.1}$$

with Λ_R being a scale of reference. We define the renormalized mass

$$m_R^2 = \frac{\Lambda_K^2}{\chi^{(2)}(\beta_c - \lambda^{-K} u)}, \tag{12.2}$$

where $\chi^{(2)}\left(\beta_c - \lambda_1^{-K} u\right)$ means the susceptibility at $\beta = \beta_c - \lambda_1^{-K} u$. Given that

$$\lambda_1^{\gamma} = 2^{\frac{2}{D}}, \tag{12.3}$$

the dependence on the UV cutoff disappears at leading order and one obtains

$$m_R^2 = \frac{\Lambda_R^2 u^{\gamma}}{A_0^{(2)} + A_1^{(2)} u^{\Delta} \left(\frac{\Lambda_R}{\Lambda_K}\right)^{\frac{2\Delta}{\gamma}} + \mathrm{LPC} + \cdots}, \tag{12.4}$$

with the log-periodic corrections

$$\mathrm{LPC} = A_{per.}^{(2)} \cos\left(\omega\left(\ln u + \frac{2}{\gamma} \ln\left(\frac{\Lambda_R}{\Lambda_K}\right)\right) + \phi^{(2)}\right). \tag{12.5}$$

In the infinite cutoff limit ($K \to \infty$), the subleading corrections disappear. On the other hand, the LPC do not, and we are in the presence of a limit cycle with a cutoff dependence quite similar to [126, 127]. Strictly speaking, the infinite cutoff limit does not exist, however, for practical purpose, the effects of the oscillations are so small that it introduces uncertainties that are smaller than the accuracy with which we establish the universality.

We could now define renormalized coupling constants in cutoff units using the higher order $\chi^{(q)}$ just as we have done in equation (12.2) for the renormalized mass. However, it is

Table 3. Universal values of $U^{(2l)\star}$.

$2l$	$U^{(2l)\star}$
4	1.505 871
6	18.107 22
8	579.970
10	35 653.8
12	$3.577\,69 \times 10^{6}$
14	$5.317\,63 \times 10^{8}$
16	$1.097\,20 \times 10^{11}$
18	$3.000\,25 \times 10^{13}$
20	$1.049\,98 \times 10^{16}$

usually more convenient [128] to use the value of these couplings in units of the renormalized mass. We thus consider dimensionless couplings of the form

$$U^{(q)} \propto \chi^{(q)}\left(\beta_c - \lambda_1^{-L} u\right) m_R^{q(1+D/2)-D}. \tag{12.6}$$

The fact that the quantity is dimensionless implies that the UV cutoff dependence disappears and we are left with a quantity proportional to the ratios of susceptibilities defined in equation (11.6). For $D = 3$, we define

$$U^{(2l)} \equiv \lim_{\beta \to \beta_c} (-1)^{l+1} \chi^{(2l)} (\chi^{(2)})^{(3-5l)/2} \beta^{3(1-l)/2}. \tag{12.7}$$

In subsection 11.4, it has been argued that along the unstable manifold, this quantity is approximately universal. Indeed by taking the limit $\beta \to \beta_c$, the flow starts on the stable manifold but then ends up on the unstable manifold and the $U^{(2l)}$ should be universal in the approximation where the very small log-periodic oscillations are neglected. This approximate universality means that once we have picked the renormalized mass, all the other renormalized couplings are completely fixed.

A counterpart of this discussion for the LPA of ERGE can be found in sections 2.10 and 3.4 of [8] where a discussion of the various continuum limits that can be constructed near the Gaussian fixed point can also be found. Field theoretical approaches of IR stable trajectories are also discussed in [24–26]. It would be interesting to see how the notions developed in these articles (for instance the 'large river effect') can be used for the HM.

12.2. Numerical estimates of the universal ratios

These expectations have been checked numerically in [129] by calculating the $U^{(2l)\star}$ for four different measures. The results were consistent with universality with six or seven significant digits. The results are given in table 3 with uncertainties of order 1 in the last printed digit.

Various fits of the asymptotic behaviour were performed in [129] and it was concluded that the leading growth is consistent with

$$U^{(q)\star} \approx q!. \tag{12.8}$$

This is illustrated in figure 8 where $\ln(U^{(2q+2)\star}/U^{(2q)\star})$ is plotted versus $\ln(2q)$. A factorial growth as in equation (12.8) would imply a straight line with slope 2, which is very close to the slope 2.1 of the linear fit in figure 8.

This factorial growth is similar to what is found in [130–133] for other models studied in the context of multiparticle production. Note that the generating function of the connected $2l$-points function has a $1/(2l)!$ factor at order $2l$ (see equation (2.8)) which means that the

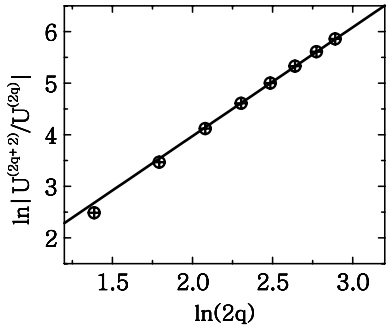

Figure 8. $\ln(U^{(2q+2)\star}/U^{(2q)\star})$ versus $\ln(2q)$. The straight line has a slope 2.11.

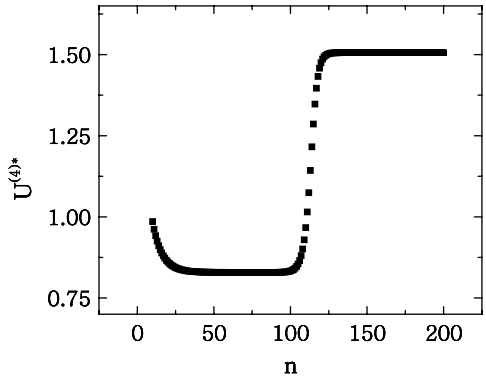

Figure 9. $U^{(4)\star}$ estimates as the number of iterations n increases.

expansion of the generating function of the connected functions in powers of an external field has a finite radius of convergence.

12.3. Other universal ratios

The $U^{2r\star}$ were calculated by picking an arbitrary measure, finding β_c corresponding to that measure and then calculating R_n until the ratios stabilize with sufficient accuracy. In this process, it can be noted that when $R_n \simeq R^\star$, typically after 20 or 30 iterations, the U^{2r} temporarily stabilize at different values than the final ones. We call these temporary values $\bar{U}^{2r\star}$. This is illustrated in figure 9 for $r = 2$. This figure shows the importance of going sufficiently far from the nontrivial fixed point to get the correct answer. Similar behaviour with temporary plateaus can be found for the LPA of ERGE in [26]. $\bar{U}^{2r\star}$ can be calculated from the numerical coefficients of $R^\star$. For instance,

$$\bar{U}^{4\star} \simeq 24 \times (0.053\,537 - 0.358\,71^2/2)/(2 \times 0.35871)^{7/2} \simeq 0.8287 \qquad (12.9)$$

This number was calculated by expanding the logarithm of equation (7.9) and then fixing the normalizations as in equation (12.7). A few other values are given in table 4. They are clearly different from the $U^{2r\star}$. From values for r up to 10, their growth seems consistent with a simple factorial.

Table 4. Universal values $\bar{U}^{(2l)\star}$.

$2l$	$\bar{U}^{(2l)\star}$
4	0.828 719
6	4.177 57
8	49.3335
10	1033.20

12.4. The critical potential of the symmetric phase

Following section 2, the effective potential can now be expressed as a function of $\chi^{(2)}$, ϕ_c and the $U^{2r\star}$. To simplify the notations, we define $m \equiv (\chi^{(2)})^{-1/2}$, the renormalized mass in cutoff units. For $D = 3$, the effective potential reads

$$V_{\rm eff}(\phi_c) = m^3 F(\phi_c m^{-1/2}), \tag{12.10}$$

with

$$F(x) = \sum_{r=1}^{\infty} \frac{f_{2r}}{2r!} x^{2r}, \tag{12.11}$$

and $f_2 = 1$, $f_4 = -U^{4\star}$, $f_6 = 10(U^{4\star})^2 - U^{6\star}$ etc... Equations (7.15) and (12.10) have the same form. The correspondence is $m \leftrightarrow \ell^{-n_{\max}}$. If we calculate the universal ratios, the parameter m or $\ell^{-n_{\max}}$ disappears, and we obtain numerical values $\bar{U}^{(2l)\star}$ for equation (7.15) and the distinct values $U^{(2l)\star}$ for equation (12.10). Consequently, the function F is distinct from the function U in equation (7.15).

In [129], rescaling in both coordinates was applied in order to compare with a parametrization introduced by Campostrini, Pelissetto, Rossi and Vicari [134] where the effective potential is expressed in terms of a universal function

$$A(z) = z^2/2 + z^4/24 + \sum_{l \geqslant 3} \frac{r_{2l}}{(2l)!} z^{2l}. \tag{12.12}$$

We have

$$r_{2l} = f_{2r}/f_4^{r-1}, \tag{12.13}$$

and these quantities can be trivially re-expressed in terms of the $U^{2r\star}$, for instance,

$$r_6 = 10 - \frac{U^{(6)\star}}{(U^{(4)\star})^2}. \tag{12.14}$$

Numerical values in table 5 are not very far from those calculated for nearest neighbour models. Approximate relations among the r_{2l} [135, 134] were also checked in [129]. The first four coefficients of the field expansion are positive, and there is no convexity issue near the origin. The discussion of the convexity at arbitrary field strength requires some understanding of the radius of convergence of the expansion. The convexity of the effective potential has been demonstrated for ERGE, using spectral representations for the RG flows, in [136] where a connection between this issue and the finite time singularity discussed above in subsection 9.5 is also drawn.

Table 5. Universal values of r_{2l} calculated numerically and compared to values obtained for the nearest neighbour Ising universality class.

2 1	r_{2l}	r_{2l} for other models
6	2.014 9752	2.048(5) [134]
8	2.679 529	2.28(8) [134]
10	−9.601 18	−13(4) [134]
12	10.7681	20(12) [137]
14	763.062	560(370) [137]

Table 6. γ, Δ and β_c/N for $N = 1, \ldots, 20$.

N	γ	Δ	β_c/N
1	1.299 140 73	0.425 946 859	1.179 030 170
2	1.416 449 96	0.475 380 831	1.236 763 288
3	1.522 279 70	0.532 691 965	1.275 794 011
4	1.608 728 17	0.590 232 008	1.302 790 391
5	1.675 510 51	0.642 369 187	1.322 083 069
6	1.726 177 03	0.686 892 637	1.336 351 901
7	1.764 798 63	0.723 880 426	1.347 244 235
8	1.794 692 74	0.754 352 622	1.355 791 342
9	1.818 271 05	0.779 508 505	1.362 657 559
10	1.837 222 91	0.800 424 484	1.368 284 407
11	1.852 726 36	0.817 977 695	1.372 974 325
12	1.865 610 92	0.832 855 522	1.376 940 318
13	1.876 469 98	0.845 589 221	1.380 336 209
14	1.885 735 62	0.856 588 705	1.383 275 590
15	1.893 728 12	0.866 171 682	1.385 844 022
16	1.900 689 03	0.874 586 271	1.388 107 107
17	1.906 803 38	0.882 027 998	1.390 115 936
18	1.912 215 07	0.888 652 409	1.391 910 870
19	1.917 037 52	0.894 584 429	1.393 524 199
20	1.921 361 21	0.899 925 325	1.394 982 051
∞	2	1	$\frac{2-c}{2(c-1)} = 1.423\,66..$

13. The large-N limit

13.1. Calculations at finite N

If we keep the $O(N)$ symmetry unbroken, the Fourier transform of the local measure depends only on $\vec{k} \cdot \vec{k} \equiv u$. Here $\vec{k}$ is a source conjugated to the local field variable $\vec{\phi}$. Replacing k by u and the second derivative by the N-dimensional Laplacian in equation (3.12), we obtain the RG transformation for the Fourier transform of the local measure

$$R_{n+1,N}(u) \propto \exp\left(-\frac{1}{2}\beta\left(4u\frac{\partial^2}{\partial u^2} + 2N\frac{\partial}{\partial u}\right)\right)(R_{n,N}(cu/4))^2. \tag{13.1}$$

The values of the exponents γ, Δ and the inverse critical temperature for a measure generalized Ising measure (or nonlinear sigma model measure) $\delta(\vec{\phi}\cdot\vec{\phi} - 1)$ calculated in [18] are given in table 6. To facilitate the comparison, we also display $\nu = \gamma/2$ (since $\eta = 0$ here) and $\omega = \Delta/\nu$ in table 7. The HM results coincide with the four digits given in column (2)

Table 7. ν, ω and α for $N = 1, \ldots, 20$.

N	$\nu = \gamma/2$	$\omega = \Delta/\nu$	$\alpha = 2 - 3\nu$
1	0.649 570	0.655 736	0.051 289
2	0.708 225	0.671 229	−0.124 675
3	0.761 140	0.699 861	−0.283 420
4	0.804 364	0.733 787	−0.413 092
5	0.837 755	0.766 774	−0.513 266
6	0.863 089	0.795 854	−0.589 266
7	0.882 399	0.820 355	−0.647 198
8	0.897 346	0.840 648	−0.692 039
9	0.909 136	0.857 417	−0.727 407
10	0.918 611	0.871 342	−0.755 834

of tables 3 (for ν) and 4 (for ω) in [101] for the Polchinski equation. They coincide with the six digits for ν given in the line $d = 3$ of table 8 of [14] for the HM with $N = 1, 2, 3, 5$ and 10. As in the case $N = 1$ discussed before, we found discrepancies of order 1 in the fifth digit of ν and slightly larger for ω with the values found in table 1 of [57]. For $N = 1$, the same discrepancy can be found in [17, 19]. Our estimated errors are of order 1 in the 9th digit. For $N = 1$, this is confirmed by an independent method [56]. For $N = 2, 3, 5$ and 10, this is confirmed up to the sixth digit [14]. Consequently, a discrepancy in the 5th digit cannot be explained by numerical errors. It seems clear that the two models are inequivalent. Note also that for $N \geqslant 2$, α, the specific heat exponent shown in table 7 is more negative than for nearest neighbour models [23, 34].

13.2. Ma's equation

The basic RG equation in the large-N limit was first derived by Ma [138]. It can be used for conventional $O(N)$ sigma models or for the $O(N)$ version of the HM. We consider the partition function

$$Z(\vec{J}) = \prod_x \int_{-\infty}^{+\infty} d^N \phi_x \, \mathrm{e}^{-S + \sum_x \vec{J}_x \vec{\phi}_x}, \tag{13.2}$$

with

$$S = -\frac{1}{2} \sum_{xy} \vec{\phi}_x \Delta_{xy} \vec{\phi}_y + \sum_x V_o\left(\phi_x^2\right). \tag{13.3}$$

We use the notation $\phi_x^2 \equiv \vec{\phi}_x . \vec{\phi}_x$, and Δ_{xy} is a symmetric matrix with negative eigenvalues. We assume that $\sum_x \Delta_{xy} = 0$. This condition is not satisfied by the quadratic form appearing in equation (3.1). Using results of subsection 14.1, it is possible to show that a term $A^\star \sum_x \phi_x^2$ must added to βH, the non-local part of the action, in order to satisfy this condition. In order to keep the original partition function invariant, this term must be subtracted from the original potential. Consequently, the potential in equation (13.3) is related to the local measure introduced in section 3 by the relation

$$V_0(\phi^2) = -\ln W_0(\phi) - A^\star \phi^2. \tag{13.4}$$

One sees that $V_0 = 0$ for the Gaussian fixed point. Defining the rescaled potential

$$V_0(X) = N U_0\left(\frac{X}{N}\right), \tag{13.5}$$

and using a saddle point calculation of the partition function in the large-N limit, it is possible to show [138, 139] that $M^2 \equiv 2\partial V_{\rm eff}/\partial\phi_c^2$ obeys the self-consistent equation

$$2U_0'\big(\phi_c^2 + f_\Delta(M^2)\big) = M^2, \tag{13.6}$$

where $f_\Delta(M^2)$ is the one-loop integral corresponding to the quadratic form Δ and a mass term M^2. The prime denotes the derivative with respect to ϕ^2. M^2 was denoted u in subsection 9.3. For the HM, it is shown in section 8 that the function f,

$$f_{\rm HM}(z) = \sum_{n=0}^{\infty} \frac{2^{-n-1}}{2A^\star(c/2)^n + z}. \tag{13.7}$$

For comparison, for a sharp cutoff model (SCM) in three dimensions, we have

$$f_{\rm SCM}(z) = \int_{|k|\leqslant 1} \frac{{\rm d}^3k}{(2\pi)^3}\frac{1}{k^2+z}. \tag{13.8}$$

For a generalization of the Ising model (nonlinear sigma models) with a local measure $\delta(\phi^2-1)$ as in [18], the saddle point condition simplifies to $f(M^2) = 1/N$. For the HM, β_c can then be determined from the condition $f_{\rm HM}(0) = 1/N$. This implies $\beta_c = N(2-c)/(2(c-1))$. This result was verified in [18] and is consistent with table 6.

Let us consider two models, the first one with a rescaled potential U_0, an UV cutoff Λ and a quadratic form Δ and a second model with a rescaled potential $U_{0,S}$, an UV cutoff Λ/S and a quadratic form Δ_S. For $D = 3$ and in the large-N limit, the two models have the same dimensionful zero-momentum Green's functions provided that

$$U_{0,S}'(\phi^2) = S^2U_0'((\phi^2 - f_{\Delta_S}(2U_{0,S}'(\phi^2))/S + f_\Delta((2/S^2)U_{0,S}'(\phi^2))). \tag{13.9}$$

In the two cases considered above $f_\Delta = f_{\Delta_\ell} \equiv f$, and the fixed point equation becomes very simple. In addition, β will be set to 1 in the rest of this section.

Following references [138, 139], we introduce the inverse function

$$F(2U_0'(\phi^2)) = \phi^2, \tag{13.10}$$

and the function $H(z) \equiv F(z) - f(z)$. With these notations, the fixed point equation corresponding to equation (13.9) is simply

$$H(z) = SH(z/S^2). \tag{13.11}$$

For the SCM, S is allowed to vary continuously in equation (13.11) and the general solution is

$$F(z) = f_{\rm SCM}(z) + Kz^{1/2}. \tag{13.12}$$

For the HM, with $D = 3$, S can only be an integer power of $\ell = 2^{1/3}$ and the general solution has an infinite number of free parameters

$$F(z) = f_{\rm HM}(z) + \sum_q K_q z^{1/2+iq\omega}, \tag{13.13}$$

with

$$\omega \equiv \frac{3\pi}{\ln 2} \simeq 13.6, \tag{13.14}$$

and q runs over positive and negative integers. There exists a unique choice of the K_q in equation (13.13) which cancels exactly the singular part of $f_{\rm HM}$. It was shown [140] that the fixed point corresponds to

$$F^\star(z) = f_{\rm HM,reg.} = \frac{1}{4A^\star}\sum_{l=0}^{\infty}\left(\frac{-z}{2A^\star}\right)^l \frac{1}{1-c^{2l-1}}. \tag{13.15}$$

This expansion has a radius of convergence $2A^\star c^2 \simeq 2.7024$ for $\beta = 1$.

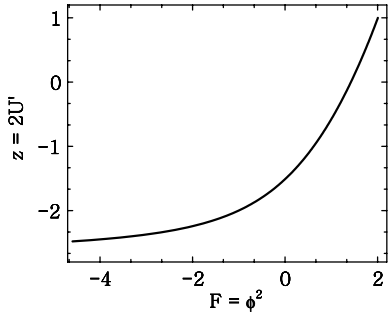

Figure 10. z versus $F^{\star}_{\rm HM}(z)$.

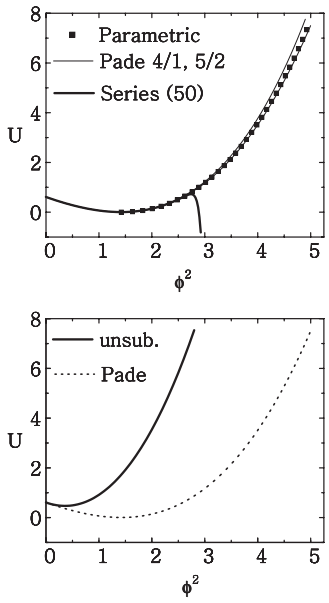

Figure 11. $U_0^{\star}(\phi^2)$ for the HM with a parametric plot (filled squares), the series truncated at order 50 (thick solid line) and Padé approximants [4/1] (thin line slightly above the squares) and [5/2] (thin line closer to the squares). The constant has been fixed in such a way that the value at the minimum is zero (upper graph). In the lower graph, we have added $A^{\star}\phi^2$ to the [5/2] Padé.

13.3. Singularities of the critical potential $U_0^{\star}$

Following [140], we can use equation (13.15) to define $F(z)$ on the negative real axis. We remind that F plays the role of ϕ^2. As we move towards more negative values of z, F becomes zero within the radius of convergence of the expansion. The situation is illustrated in figure 10. Numerically, $F^{\star}(-1.5107\ldots) = 0$. We then re-expand the series about that value of z (which corresponds to $F = \phi^2 = 0$) and invert it. The resulting series is an expansion of $2U_0^{\star}{}'$ in ϕ^2. After integration, and up to an arbitrary constant u_0, we obtain a Taylor series for the critical potential $U_0^{\star}$. We denote the expansion as

$$U_0^{\star}(\phi^2) = \sum_{n=0}^{\infty} u_n (\phi^2)^n. \tag{13.16}$$

The absolute value of the coefficients appears to grow at an exponential rate. Linear fits suggest a radius of convergence of order 2.5. The signs follow the periodic pattern $++--$. This suggests singularities along the imaginary axis. This analysis is confirmed by an analysis [140] of the poles of Padé approximants. As ϕ^2 exceeds the critical values estimated in the previous section, the power series is unable to reproduce the expected function $U_0^\star$. The situation is illustrated in figure 11. For comparison, we have also added $A^\star\phi^2$ to $U_0^\star$ in order to undo the subtraction of equation (13.4) and obtain a function that has a simple relation with the effective potential as in equation (7.15). Again, the resulting function appears to be convex. The numerical values of $U_0^\star$ in figure 11 have been calculated using a parametric representation discussed in [140]. Finite radius of convergence has been observed in expansions in the power of the fields based on ERGE in the LPA [19, 141]. More generally, it would be interesting to compare the large-N limit for the HM and in the ERGE approach as discussed, for instance, in [101, 142–146].

13.4. Open problems

The phase diagram of $3D$ models with a conventional kinetic term shows interesting features such as a line of tricritical points ending at the so-called BMB point [139, 147, 148]. A similar study should be done for the HM. A general interpretation of the complex singularities of the critical potential would be interesting. From the numerical values of the exponents, it is possible to estimate the low-order coefficients of the $1/N$ expansion [149]. The method of calculation of the coefficients using numerical values was developed with the Sterling series for which we were able to calculate the first seven coefficients accurately. These results are consistent with the hypothesis that the $1/N$ expansions considered are asymptotic but Borel summable (no indication for poles on the positive real axis).

14. The improvement of the hierarchical approximation

In this section, we describe the HM as a spin model on the 2-adic line. The main motivation for this new way to look at the model is that it suggests a way to modify the model in order to approximate nearest neighbour models in D dimensions.

14.1. Scalar models on ultrametric spaces

It is possible to reformulate the HM as a scalar model on the 2-adic line [65]. We give here a presentation [66] that does not require a detailed knowledge of the p-adic numbers. We will rewrite the Hamiltonian of the HM given in equation (3.1) using a function $v(x, y)$ which specifies the level l at which x and y start to differ. More precisely, if x and y are distinct, $v(x, y) = l$ when $x_m = y_m$ for all m such that $n \geqslant m > l > 0$ and $x_l \neq y_l$. At coinciding arguments, we define $v(x, x) = 0$. Referring to figure 1, we can see that $2^{v(x,y)}$ is the size of the smallest block containing both x and y. H can then be rewritten as

$$H = -\frac{1}{2}\left(\sum_{x,y} \mathbf{K}_{xy}\phi_x\phi_y + \mathbf{L}\sum_x \phi_x^2\right), \tag{14.1}$$

where

$$\mathbf{K}_{xy} = \begin{cases} \left(\left(\frac{c}{4}\right)^{v(x,y)} - \left(\frac{c}{4}\right)^{n+1}\right)\left(1 - \frac{c}{4}\right)^{-1} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \tag{14.2}$$

and

$$\mathbf{L} = \left(\left(\frac{c}{4}\right) - \left(\frac{c}{4}\right)^{n+1}\right)\left(1 - \frac{c}{4}\right)^{-1}. \tag{14.3}$$

As made clear by the above equations, the strength of the interaction between two fields ϕ_x and ϕ_y depends only on the value of $v(x, y)$. Consequently, the invariance of $v(x, y)$ under a group of transformation implies the invariance of H under corresponding transformations.

For the reader familiar with the p-adic numbers, The function $2^{v(x,y)}$ is a regularized version of the 2-adic distance. Namely,

$$2^{v(x,y)} = \begin{cases} |x - y|_2 & \text{if } |x - y|_2 > 1 \\ 1 & \text{if } |x - y|_2 \leqslant 1. \end{cases} \tag{14.4}$$

The 2-adic norm satisfies a relation stronger than the triangle inequality, namely

$$|x + y|_2 \leqslant \text{Max}(|x|_2, |y|_2). \tag{14.5}$$

Normed spaces for which this stronger inequality holds are called ultrametric spaces. These concepts are explained in more detail for instance in [61, 63].

In order to describe the invariance of $v(x, y)$, we associate with the sequence of 0's and 1's $x_n, \ldots, x_1$, introduced in section 3 for 2^n sites, a rational number of the form

$$x = \sum_{m=1}^{n} x_m 2^{-m}. \tag{14.6}$$

The reason for this 'inversion' is that $|2|_2 = 2^{-1}$, and similarly in momentum space, the largest shell will correspond to odd integers, the next shell by the multiple of 2 of these odd integers and so on. If two numbers x and y have this form then $x + y$ can also be written in this form provided that we drop the integer part of the sum. Equivalently, we can write $x = q/2^n$ and $y = r/2^n$ with q and r integers between 0 and $2^n - 1$ and add q and r *modulo* 2^n. Since the integers *modulo* 2^n form an additive group, the set of fractions associated with the sites form a group for the addition modulo 1. The odd integers *modulo* 2^n form a multiplicative group. In the limit $n \to \infty$, the group is called the 2-adic units. We can pick a canonical form for the representatives of such integers as

$$u = 1 + 2z, \tag{14.7}$$

where z is a positive integer between 0 and $2^{n-1} - 1$. Obviously, if x has the form of equation (14.6) then ux has also this form after discarding its integer part.

We are now in a position to define a group of transformation acting on the fractions associated with the sites. If x and a have the form of equation (14.6) and u has the form of equation (14.7), we define a transformation of x depending on a and u and denoted $x[u, a]$ which reads

$$x[u, a] = ux + a, \tag{14.8}$$

where the rhs is understood *modulo* 1. It is clear that these transformations form a (non-Abelian) group that we could call the Poincaré group of the HM. We can interpret $x[0, a]$ as a translation and $x[u, 0]$ as a rotation like in the usual Poincaré groups. In that sense, this is a 'global' group of transformation. This is in contrast with the symmetries noted by Dyson [5] which consists in interchanging $x_m \ldots x_{l+1} 1 x_{l-1} \ldots x_1$ and $x_m \ldots x_{l+1} 0 x_{l-1} \ldots x_1$ 'locally' which are discussed in section 3. It is possible to prove that

$$v(x[u, a], y[u, a]) = v(x, y). \tag{14.9}$$

Since $\mathbf{K}_{xy}$ depends only on $v(x, y)$, this implies that

$$\mathbf{K}_{xy} = \mathbf{K}_{x[u,a]y[u,a]} \tag{14.10}$$

that H is invariant under the transformation

$$\phi_x \longrightarrow \phi_{x[u,a]}. \tag{14.11}$$

We can then prove that

$$\langle \phi_{x[u,a]} \phi_{y[u,a]} \rangle = \langle \phi_x \phi_y \rangle. \tag{14.12}$$

14.2. Improvement of the hierarchical approximation

In this subsection, we follow [20, 21]. We consider a Gaussian model on a finite one-dimensional lattice with 2^n sites. The Fourier modes of the scalar field are denoted by Φ_k, where $\Phi_k^\star = \Phi_{-k}$ and k are integers used to express the momenta in $\frac{2\pi}{2^n}$ units. These integers are understood modulo 2^n in the following (periodicity in momentum space). The action reads

$$S = \frac{1}{2^{n+1}} \sum_{k=1}^{2^n} g(k) \Phi_k \Phi_{-k}, \tag{14.13}$$

where $g(k)$ is even, real and positive.

We proceed in three steps. First, we relabel the momenta in a way which is convenient to read their 'shell' assignment as in Wilson [3]. This relabelling allows us to introduce a group of transformation whose orbits are precisely these shells. The group is the multiplicative group of the 2-adic units and its representations are known [150]. Completeness can be used to expand the kinetic term, i.e., the function $g(k)$, for each of the shells. This solves the bookkeeping problem. Nicely enough, the classification of the representations of the group mentioned above comes with an index indicating its resolution power (called the degree of ramification). This naturally provides the successive orders of our perturbative expansion. We then show that if we only retain the trivial representation in the expansion, we obtain the hierarchical approximation. In this limit, the group of transformation is a symmetry of the action which can be identified with the symmetry group of the HM mentioned above.

We can relabel the momenta k. For this purpose, we use a set of orthonormal functions which is a discrete version of the Walsh system [151]. We first define

$$\Psi_0(k) \equiv \begin{cases} 1 & \text{if } k = -2^{n-2}+1, \ldots, 2^{n-2}-1 \\ \omega & \text{if } k = 2^{n-2} \\ \omega^{A^\star} & \text{if } k = -2^{n-2} \\ 0 & \text{otherwise,} \end{cases} \tag{14.14}$$

and

$$\Psi_1(k) \equiv 1 - \Psi_0(k), \tag{14.15}$$

with the notation $\omega \equiv \frac{1+i}{2}$. For a given integer $a = a_0 + a_1 2^1 + a_2 2^2 + \cdots + a_{n-1} 2^{n-1}$ with $a_l = 0 \text{or} 1$, we define

$$f_a(k) \equiv \prod_{l=0}^{n-1} \Psi_{a_l}(2^l k). \tag{14.16}$$

It is clear that $f_a^\star(k) = f_a(-k)$ and we can check that

$$\sum_k f_a(k) f_b^\star(k) = \delta_{a,b}. \tag{14.17}$$

A more detailed analysis shows that $f_a(k)$ is non-zero only when $k = \pm k[a]$ for a function $k[a]$ which will be specified. More precisely, it is possible to write

$$f_a(k) = \omega \delta_{k,k[a]} + \omega^\star \delta_{k,-k[a]}. \tag{14.18}$$

This relation *defines* a one-to-one map $k[a]$. In the following, a will also be treated as an integer modulo 2^n. We can now expand

$$\Phi_k = \sum_{a=1}^{2^n} c_a f_a(k). \tag{14.19}$$

We can then rewrite

$$S = \frac{1}{2^{n+1}} \sum_{a=1}^{2^n} \tilde{g}(a) c_a^2, \tag{14.20}$$

where $\tilde{g}(a) \equiv g(k[a])$. By construction, the c_a are real field variables. For convenience, we shall also use their complex form

$$\sigma_a \equiv \frac{1}{2}(c_a + c_{-a}) + \frac{\mathrm{i}}{2}(c_a - c_{-a}). \tag{14.21}$$

We can now explain the correspondence between this relabelling and Wilson's cell decomposition. Clearly, if $a_0 = 1$, $f_a(k)$ is supported in the high momentum region. More precisely, the 0th shell, i.e, the one integrated first in the RG procedure, consists in configurations which can be expanded in terms of the $f_{1+a_1 2+\dots}(k)$. Similarly, the modes corresponding to the lth shell are made out of the $f_{2^l+a_{l+1}2^{l+1}+\dots}(k)$.

14.3. The hierarchical approximation and its systematic improvement

In the previous section, we have introduced new field variables c_a corresponding to the lth shell when a can be divided by 2^l but not by 2^{l+1}. This property is not affected if a is multiplied (modulo 2^n) by any odd number. As explained in subsection 14.1, odd numbers form an Abelian group with respect to the multiplication modulo 2^n. The orbit of this group within the integers modulo 2^n is precisely the sets of numbers that we have put in correspondence with the shells. The representations of this group have been studied and classified [150]. In order to use this classification, we have to embed the labels introduced above and denoted by a, in the 2-adic integers. When a can be divided by 2^l but not by 2^{l+1}, we say that the 2-adic norm, noted $|a|_2$, is 2^{-l}. In the infinite volume limit, or in other words when n tends to infinity, the multiplicative group of the odd numbers is the 2-adic units. The representations of this group will be denoted by Π_s. This means that if u_1 and u_2 are 2-adic units, then $\Pi_s(u_1 u_2) = \Pi_s(u_1)\Pi_s(u_2)$. The label s specifies the representation in a way which will be explained below.

It is easy to construct explicitly the representations Π_s. A 2-adic unit can be written [152] as $u = \pm\mathrm{Exp}(4z)$, where z is a (2-adic) integer and Exp the 2-adic exponential. $\Pi_s(u)$ is even or odd under multiplication by -1. On the other hand, z is an additive parametrization and the z dependence of Π_s will be of the form $\mathrm{e}^{\mathrm{i}\frac{2\pi z q}{2^r}}$, where q is an odd integer and r is a positive integer. Taibleson [150] calls $r+2$ the degree of ramification. In summary, the label s is a short notation for the parity, r and q an odd integer modulo 2^r.

We can now use these representations to expand the kinetic term function $\tilde{g}(a)$ in each shell. For a given shell l, the a have the form $2^l u$ (so $|a|_2 = 2^{-l}$) and we can write

$$\tilde{g}(2^l u) = \sum_s g_{l,s}\, \Pi_s(u). \tag{14.22}$$

At finite volume, i.e., at finite n, the units are understood modulo 2^{n-l}, and consequently the sum over the representations s is restricted to $r \leqslant n - l - 2$. The numerical coefficients $g_{l,s}$ are easily calculable using the orthogonality relations among the representation.

The hierarchical approximation is obtained by retaining only the trivial representation in the expansion equation (14.22). In this approximation, and using the definition introduced in equation (14.21), the action reads

$$S = \frac{1}{2^{n+1}} \sum_{l=0}^{n-1} g_{l,+,0} \sum_{a:|a|_2=2^{-l}} \sigma_a \sigma_{-a}. \tag{14.23}$$

After a Fourier transform, we obtain a hierarchical model having the general form (with arbitrary b_l couplings at level l) (see section 6).

The classification of the representations of the 2-adic units suggests that we improve the hierarchical approximation by taking into account the additional terms in equation (14.22) *order by order in the degree of ramification*. Intuitively, this corresponds to the fact that the degree of ramification measures the 'power of resolution' of the representation. Numerically, this works reasonably well: in a simple example, the coefficients become smaller as the degree increases [20].

The actions written above take a more familiar form after Fourier transformation in a. In general, the lth momentum shell is responsible for interactions among the averages of the Fourier transformed fields inside boxes of size 2^l. In the hierarchical approximation, the interactions depend only on the position of these boxes inside boxes of size 2^{l+1}. When the corrections are introduced up to $r = r_{\max}$, the interactions depend on the position inside boxes of size $2^{l+2+r_{\max}}$.

14.4. The improvement of the hierarchical approximation as a symmetry breaking problem

It is important to realize that in the hierarchical approximation, S is invariant under the transformation

$$\sigma_a \longrightarrow \sigma_{ua} \tag{14.24}$$

for any odd number u. When the other terms of the expansion are incorporated, this symmetry is broken by each term in a definite way. This allows us to use Ward identities techniques. We discuss the simplest case below.

Suppose we want to calculate the two-point function using the perturbative expansion described in the previous section. First we use the new variables c_a and the inverse of the map $k[a]$ defined in section 2 to write

$$\langle \Phi_k \Phi_{-k} \rangle = \langle c^2_{a[k]} \rangle. \tag{14.25}$$

In the hierarchical approximation, the value of this expression depends only on the momentum shell specified by $|a|_2$. In other words,

$$\langle c^2_{ua} \rangle_0 = \langle c^2_a \rangle_0, \tag{14.26}$$

where $\langle \cdots \rangle_0$ means that the quantity is evaluated, at order 0, or in other words, in the hierarchical approximation.

Suppose that we now include a correction $\delta\tilde{g}(a)$ to the approximation of $\tilde{g}(a)$. Then in first order in this perturbation, we recover the a momentum dependence within the shells given by

$$\langle c^2_{ua} \rangle_1 = \langle c^2_a \rangle_1 - \frac{1}{2^{n+1}} \sum_b (\delta\tilde{g}(ub) - \delta\tilde{g}(b)) \langle c^2_b c^2_a \rangle_0. \tag{14.27}$$

In the model considered here, the $\langle c^2_b c^2_a \rangle_0$ contribution can be evaluated straightforwardly, and we can check that we recover the first term in the expansion of $(1/\tilde{g}(ua)) - (1/\tilde{g}(a)))$. The

important point is that the corrections are evaluated using the unperturbed action. It is clear that similar methods can be used for higher point functions and in the interacting case.

An extension to D dimension can be constructed easily by noting the approximate correspondence between the integration over successive shells and the block-spin method. On a D-dimensional cubic lattice, we can decompose the block-spin procedure into D steps (one in each directions).

14.5. Other applications

The 2-adic formulation of the HM can be used for other purposes. For instance, it is possible to understand the absence of certain diagrams in the approximate recursion formula [4]. The lack of diagrams with three lines having large momenta coming out of a vertex can be understood from the fact that the sum of two 2-adic integers with 2-adic norm 1 is a 2-adic integer with a strictly lower norm. In simpler words, the sum of two odd numbers is even. 2-adic analysis was also used to study the Symanzik representation of the HM [153]. The basic ingredient [64] being that the quadratic form in H defines a random walk with a Hausdorff dimension $2/D$.

15. Models with approximate supersymmetry

In subsections 7.6 and 12.1, we made clear that the continuum limit requires a fine tuning and that this procedure may be seen as unnatural. This feature is clearly related to the existence of an unstable direction. In four dimensions, there are several ways to get rid of the unstable directions and limit the flow to marginal directions. One possibility is to impose a gauge symmetry that forbids a mass term for the gauge fields. Another possibility is to introduce new degrees of freedom with opposite statistics that partially cancel the quantum fluctuations. This possibility has been exploited in the perturbative treatment of supersymmetric models. In the following, we follow this second idea and construct models with approximate supersymmetry. Other hierarchical models involving fermions can be found in the literature [154, 155].

We consider a free action for N massless scalar fields $\phi_x^{(i)}$ and fermion fields $\psi_x^{(i)}$ and $\bar{\psi}_x^{(i)}$:

$$S_{\text{free}} = \frac{1}{2}\sum_{x,y,i} \phi_x^{(i)} D_{xy}^2 \phi_y^{(i)} + \sum_{x,y,i} \bar{\psi}_x^{(i)} D_{xy} \psi_y^{(i)}, \tag{15.1}$$

where x and y run over the sites and i from 1 to N. The $\psi_x^{(i)}$ and $\bar{\psi}_x^{(i)}$ are Grassmann numbers integrated with a measure

$$\int \prod_{x,i} \mathrm{d}\psi_x^{(i)}\, \mathrm{d}\bar{\psi}_x^{(i)}. \tag{15.2}$$

We require that D_{xy}^2 has positive eigenvalues and that we can write

$$D_{xy}^2 = \sum_z D_{xz} D_{zy}. \tag{15.3}$$

The free action $S_B^{\text{free}} + S_F^{\text{free}}$ is invariant at first order under the transformation

$$\begin{aligned} \delta\phi_x^{(i)} &= \epsilon\bar{\psi}_x^{(i)} + \psi_x^{(i)}\bar{\epsilon} \\ \delta\psi_x^{(i)} &= \epsilon \sum_x D_{xy}\phi_y^{(i)} \\ \delta\bar{\psi}_x^{(i)} &= \bar{\epsilon} \sum_x D_{xy}\phi_y^{(i)}. \end{aligned} \tag{15.4}$$

The ϵ and $\bar{\epsilon}$ are Grassmann numbers. Integration by part or Leibnitz's rule cannot be used for D_{xy}, and the order $\epsilon\bar{\epsilon}$ variations do not cancel.

We now give the explicit form for the bosonic part at finite volume:

$$S_B^{\text{free}} = -\frac{\beta_B}{2}\sum_{n=1}^{n_{\max}}\left(\frac{c_B}{4}\right)^n \sum_{x_{n_{\max}},\ldots,x_{n+1},i}\left(\sum_{x_n,\ldots,x_1}\phi^{(i)}_{(x_{n_{\max}},\ldots,x_1)}\right)^2 + \frac{\beta_B c_B}{2(2-c_B)}\sum_{x_{n_{\max}},\ldots,x_{n+1},i}\left(\phi^{(i)}_{(x_{n_{\max}},\ldots,x_1)}\right)^2, \tag{15.5}$$

with $c_B = c = 2^{1-2/D}$. The fermionic part reads

$$S_F^{\text{free}} = -\beta_F\sum_{n=1}^{n_{\max}}\left(\frac{c_F}{4}\right)^n \sum_{x_{n_{\max}},\ldots,x_{n+1},i}\left(\sum_{x_n,\ldots,x_1}\bar{\psi}^{(i)}_{(x_{n_{\max}},\ldots,x_1)}\right)\left(\sum_{x_n,\ldots,x_1}\psi^{(i)}_{(x_{n_{\max}},\ldots,x_1)}\right) \tag{15.6}$$

$$+\frac{\beta_F c_F}{2-c_F}\sum_{x_{n_{\max}},\ldots,x_{n+1},i}\bar{\psi}^{(i)}_{(x_{n_{\max}},\ldots,x_1)}\psi^{(i)}_{(x_{n_{\max}},\ldots,x_1)}, \tag{15.7}$$

with $c_F = 2^{1-1/D}$. We have the simple relation $c_B/2 = (c_F/2)^2$. Using the techniques explained in [66], one can show that the fermionic operator is the square root of the bosonic operator (see equation (15.3)) provided that

$$\frac{\beta_F c_F}{2-c_F} = \left(\frac{\beta_B c_B}{2-c_B}\right)^{\frac{1}{2}}. \tag{15.8}$$

We can introduce local interactions. The Grassmann nature of the fermionic fields restricts severely the type of interactions allowed. For instance, for one flavour ($N = 1$), the most general bosonic local measure is

$$\mathcal{W}(\phi,\psi,\bar{\psi}) = W(\phi) + \psi\bar{\psi}A(\phi). \tag{15.9}$$

For convenience, we can reabsorb the local quadratic terms in the local measure. In the following, $W(\phi)$ will take the Landau–Ginzburg (LG) form:

$$W(\phi) \propto \exp\left(-\left(\frac{\beta_B c_B}{2(2-c_B)} + \frac{1}{2}m_B^2\right)\phi^2 - \lambda_B\phi^4\right). \tag{15.10}$$

If the two functions W and A are proportional, the fermionic degrees of freedom decouple. The renormalization group transformation takes the form

$$W \to 2A \star W \tag{15.11}$$

$$A \to 2\beta_F A \star W + \left(\frac{4}{c_F}\right) W \star W, \tag{15.12}$$

where the $\star$ operation is defined as

$$(A \star B)(\phi) \equiv \mathrm{e}^{\frac{\beta_B}{2}(\phi^2)}\int \mathrm{d}\phi' A\left(\frac{(\phi 2c_B^{-\frac{1}{2}} - \phi')}{2}\right) B\left(\frac{(\phi 2c_B^{-\frac{1}{2}} + \phi')}{2}\right). \tag{15.13}$$

The introduction of a Yukawa coupling can be achieved by having $A(\phi)$ linear. Such a term breaks explicitly the Z_2 symmetry of the LG measure. Models with two flavours ($i = 1, 2$) with the type of bilinear coupling appearing in the Wess–Zumino [156] model can be

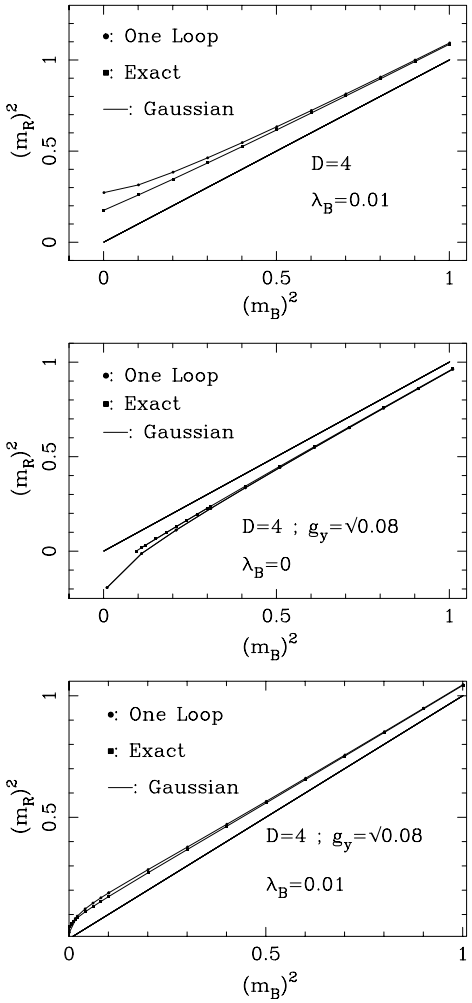

Figure 12. The renormalized mass as a function of the bare mass in a bosonic $O(2)$ model with bare quartic coupling fixed to 0.01 (top), with bare quartic coupling fixed to 0 and a Yukawa coupling equal to $\sqrt{0.08}$ (middle) and with bare quartic coupling fixed to 0.01 and a Yukawa coupling equal to $\sqrt{0.08}$ (bottom).

written as

$$\mathcal{W}(\phi^{(i)}, \psi^{(i)}, \bar{\psi}^{(i)}) = W(\phi^{(i)}) + A(\phi^{(i)})(\bar{\psi}^{(1)}\psi^{(1)} + \bar{\psi}^{(2)}\psi^{(2)}) + B(\phi^{(i)})\psi^{(1)}\psi^{(2)} - B^{\star}(\phi^{(i)})\bar{\psi}^{(1)}\bar{\psi}^{(2)} + T(\phi^{(i)})\bar{\psi}^{(1)}\psi^{(1)}\bar{\psi}^{(2)}\psi^{(2)}. \tag{15.14}$$

This is not the most general measure, however it closes under the renormalization group transformation which takes the form

$$\begin{aligned} W &\to (W \star T + A \star A + B \star B^{\star}) \equiv W' \\ A &\to \beta_F W' + \frac{4}{c_F} A \star T \\ B &\to \frac{4}{c_F} B \star T \\ T &\to \frac{8}{c_F^2} T \star T + \beta_F \frac{8}{c_F} A \star T + (\beta_F)^2 W'. \end{aligned} \tag{15.15}$$

In addition, if we impose that the function B has the following form:

$$B(\phi^{(i)}) = (\phi^{(1)} + \mathrm{i}\phi^{(2)})P((\phi^{(1)})^2 + (\phi^{(2)})^2), \tag{15.16}$$

while W, A and T are $O(2)$ invariant, the model is then invariant under the R-symmetry

$$\begin{aligned} &(\phi^{(1)} + \mathrm{i}\phi^{(2)}) \to \mathrm{e}^{\mathrm{i}\theta}(\phi^{(1)} + \mathrm{i}\phi^{(2)}) \\ &\psi^{(j)} \to \mathrm{e}^{-\mathrm{i}\frac{\theta}{2}}\psi^{(j)} \\ &\bar{\psi}^{(j)} \to \mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\bar{\psi}^{(j)}. \end{aligned} \tag{15.17}$$

We summarize three numerical calculations [157] performed for the second model with $D = 4$. First, the case where the fermions decouple from the bosons was considered. W takes the form

$$W(\phi) \propto \exp\left(-\left(\left(\left(\frac{\beta_B c_B}{2(2-c_B)}\right) + \frac{1}{2}m_B^2\right)\sum_i (\phi^{(i)})^2 + \lambda_B\left(\sum_i (\phi^{(i)})^2\right)^2\right)\right). \tag{15.18}$$

The value of m_R^2, defined as the inverse of the zero-momentum two-point function, is shown in the top part of figure 12 as a function of m_B^2. These quantities are expressed in cutoff units. For reference, we have also displayed the one-loop perturbative result and the trivial Gaussian result. One sees that the scalar self-interaction moves m_R^2 up and $m_R^2 \simeq 0.2$ when m_B^2 goes to zero. The one-loop result is quite good when m_R^2 is large enough but deteriorates when this quantity becomes smaller.

The second calculation was done in a bosonic model with a bare mass m_B and $\lambda_B = 0$ coupled to a fermion with the following couplings:

$$\begin{aligned} A &= (-1 - m_B)W \\ P &= g_y W \\ T &= ((-1 - m_B)^2 + g_y^2((\phi^{(1)})^2 + (\phi^{(2)})^2))W. \end{aligned} \tag{15.19}$$

The results are shown in the middle part of figure 12 for $g_y = \sqrt{0.08} \simeq 0.28$. One sees that the Yukawa coupling moves m_R^2 down. For $m_B^2 \simeq 0.094$, m_R becomes 0 and for smaller of m_B^2, we enter the broken symmetry phase.

Finally, the previous calculation was repeated with $\lambda_B = 0.01$ instead of 0. In perturbation theory, the one-loop quadratic divergence cancel when $m_B = 0$ and

$$8\lambda_B = g_y^2, \tag{15.20}$$

which justifies the choice of the coupling constant. The results are shown in figure 12. One sees that the Yukawa coupling in part cancels the effects of the scalar self-interaction, however, the cancellation is not as good as in the one-loop formula where m_R goes to zero when m_B^2 goes to zero. Instead, we found numerically that $m_R^2 \simeq 0.044$ when m_B^2 goes to zero. It is of course possible to fine tune g_y in order to get $m_R = 0$.

16. Conclusions

In conclusion, we have shown that the calculation of the effective potential at the nontrivial fixed point (equation (7.15)) and for a massive theory in the symmetric phase (equation (12.10)) can be performed numerically with great accuracy. The second calculation requires the ability of following the RG flows all the way from the nontrivial point to the HT attractive fixed point. For the HM, this can be accomplished numerically or by constructing the nonlinear scaling variables. The situation could be compared to a quantum mechanical problem which can be solved accurately and consistently by different methods but for which there is no closed form

solution. There remain many open problems: the construction of the scaling variables near the Gaussian fixed point and in the broken symmetry phase, the exploration of the phase diagram for models with N components, the explicit calculation of the $1/N$ expansion of the critical exponents and most importantly, the improvement of the hierarchical approximation.

The HM is a good laboratory to explore new ideas. In particular, the idea of introducing a large field cutoff [85] in order to produce a perturbative series with a better large-order behaviour or to generate a new type of RG flows when the field cutoff is lowered [158] according to the simple scheme

$$\int_{\|\phi\|<\phi_{\max}} D\phi\, \mathrm{e}^{-S} = \int_{\|\phi\|<\phi_{\max}/\xi} D\phi\, \mathrm{e}^{-S'(\xi)}. \tag{16.1}$$

Our review emphasizes the connection with the ERGE and the possibility of moving among improvements of the LPA by taking the limit $\ell \to 1$ or discretizing continuous RG flow equations. It should be reminded that before approximations are made, ERGE can be formulated several equivalent ways. For instance, the exact Polchinski equation is related to flow equations for the effective action [100, 159, 160] by a Legendre transform. However, their derivative expansions are inequivalent and there is room for optimization [93]. In contrast, for the HM, despite a programme of systematic improvement, exact equations for lattice models remain to be found. We hope that some communication can be established between these approaches in the future.

Except for section 15, most of the numerical work was done for $D = 3$. Considering models with bosons and fermions producing effects of opposite signs on the effective potential may help resolve controversial issues [161] regarding the triviality and the stability of the effective potential in $D = 4$.

Finally, it would be desirable to apply similar methods for gauge theories. In quantum chromodynamics, understanding how asymptotic freedom and confinement can be smoothly connected in a single theory amounts to constructing the renormalization group (RG) flows of the theory far away from the two fixed points of interest. In this example, it is expected that nothing dramatic takes place as we interpolate between the two regimes; however, understanding confinement in terms of the weak-coupling variables remains a challenge. Proving the existence of a mass gap in Yang–Mills theories remains one of the Clay Millennium Prize Problems. Recent efforts have made to understand the basic problems by using decimation procedures on the lattice [38, 40, 41] by using ERGE functional methods [10, 37, 39, 41, 42]. As the large hadron collider is almost in operation, a more solid understanding of gauge theories should be a priority in the theory community.

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